Progress in both all Mach number flow and well-balanced methods for the compressible Euler equations

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AMS Seminar Series, September 15, 2015, NASA Ames Research Center

where is Würzburg ?

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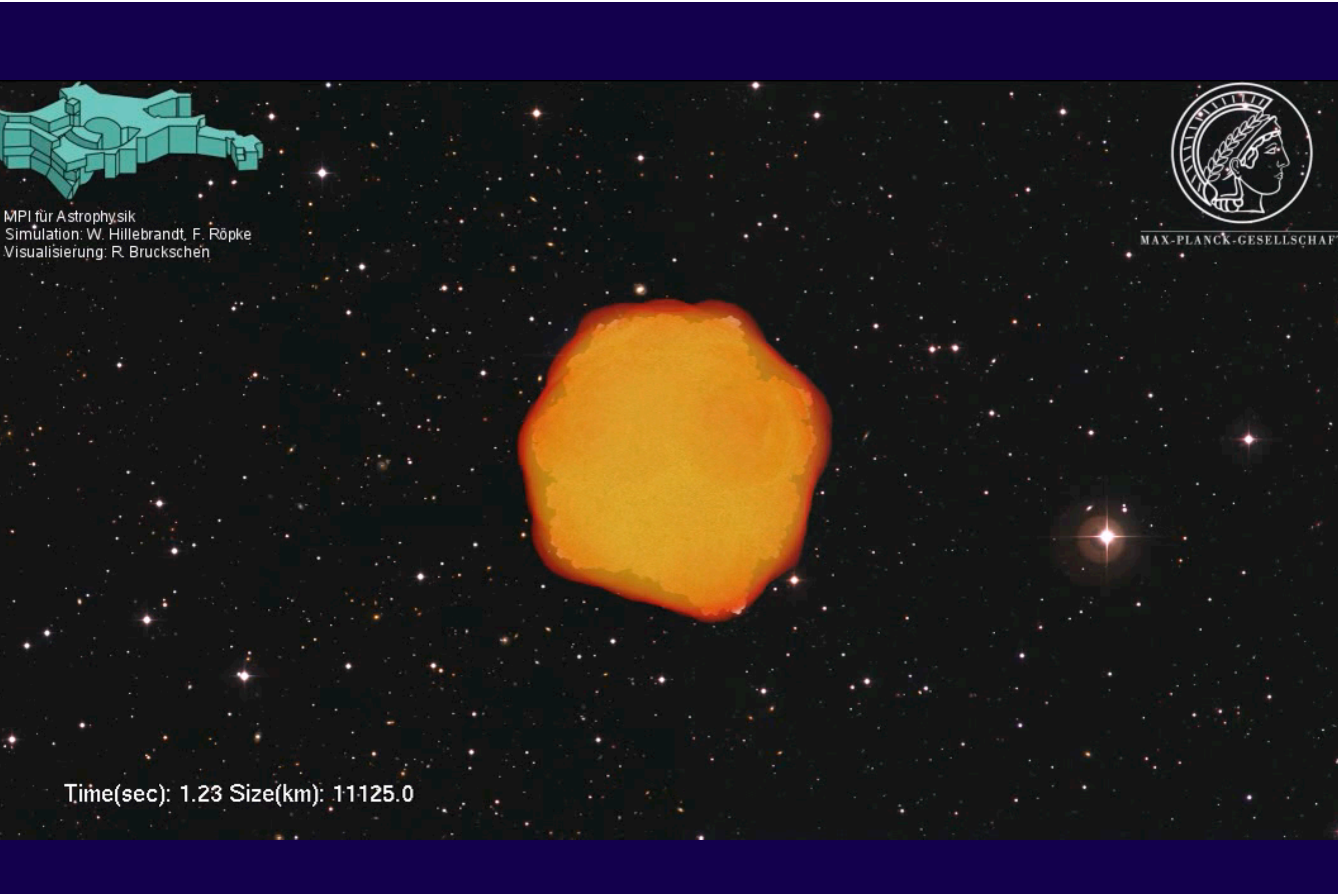
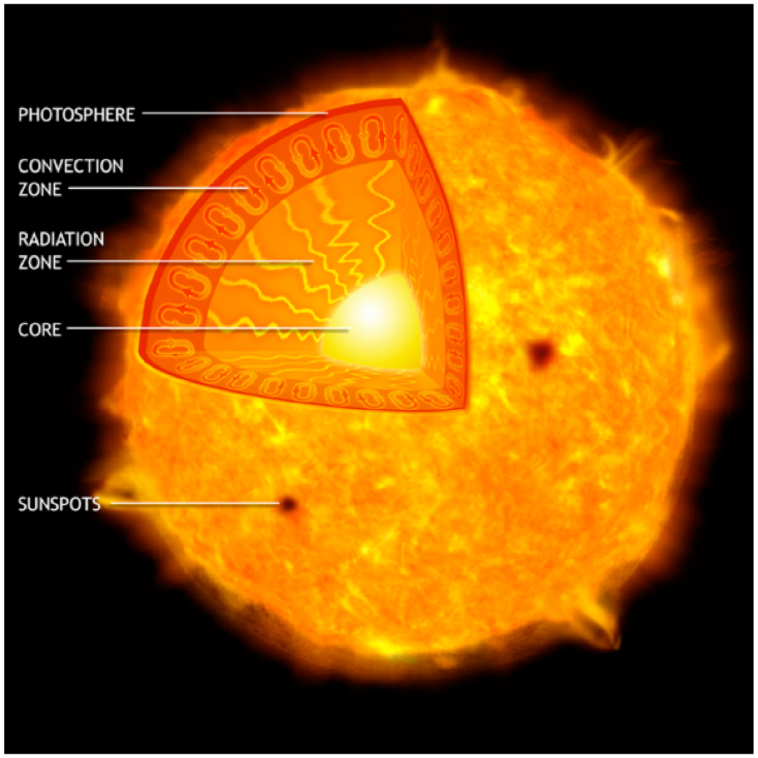
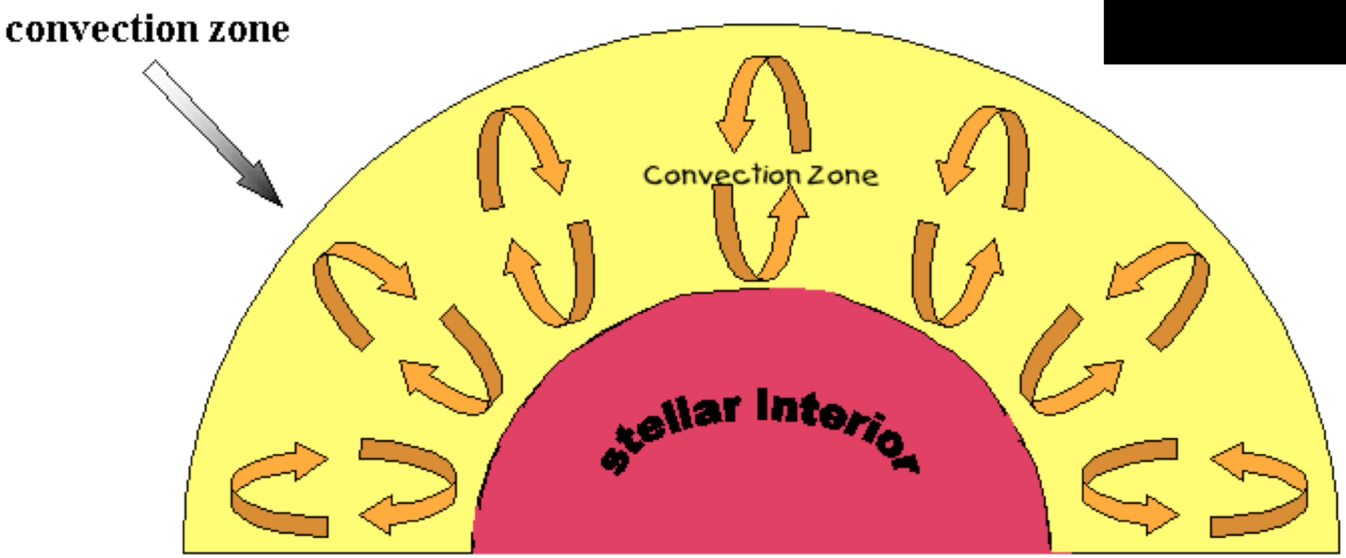
My astrophysics colleague Fritz Röpke is an expert in simulating supernovae Ia explosions. Here is a simulation of his of a supernovae Ia explosion.


MPI für Astrophysik
Simulation: W. Hillebrandt, F. Röpke
Visualisierung: R. Bruckschen
MAX-PLANCK-GESELLSCHAFT
Time(sec): 1.23 Size(km): 11125.0

- convection in stars





The Euler equations with gravity:

$$
\begin{aligned}
\partial_t \rho + \partial_i(\rho v^i) &= 0 \\
\partial_t(\rho v^j) + \partial_i(\rho v^i v^j + \delta^{ij} p) &= \rho g^j \\
\partial_t e + \partial_i(v^i(e+p)) &= \rho g^i v^j \delta_{ij}
\end{aligned}
$$

non-dimensionalize these equations:

$$
\begin{aligned}
\mathrm{Str}\,\partial_{\tilde{t}} \tilde{\rho} + \tilde{\partial}_i(\tilde{\rho}\tilde{v}^i) &= 0 \\
\mathrm{Str}\,\partial_{\tilde{t}}(\tilde{\rho}\tilde{v}^j) + \tilde{\partial}_i(\tilde{\rho}\tilde{v}^i\tilde{v}^j) + \frac{1}{\mathrm{M}^2}\delta^{ij}\tilde{\partial}_i\tilde{p} &= \tilde{\rho}\tilde{g}^j\frac{1}{\mathrm{Fr}^2} \\
\mathrm{Str}\,\partial_{\tilde{t}}\tilde{e} + \tilde{\partial}_i(\tilde{v}^i(\tilde{e}+\tilde{p})) &= \tilde{\rho}\tilde{v}^i\tilde{g}^j\delta_{ij}\frac{\mathrm{M}^2}{\mathrm{Fr}^2}
\end{aligned}
$$

obtain Mach, Froude & Strouhal numbers

set Strouhal number = 1

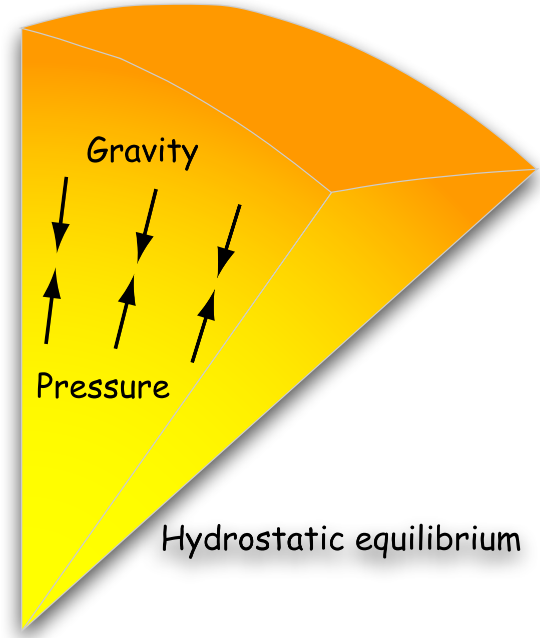
means time scale depends on length and velocity scale

set Froude number = Mach number

means internal energy ~ gravitational energy

$$\partial_x p = -\rho \partial_x \phi$$

- *1 space dim.*
- *stationary*
- *set velocity = 0*



Consider flow that is a perturbation of the hydrostatic equilibrium, the flow is then close to incompressible flow.

Thus we need a solver that

- *maintains hydrostatic equilibria well*
- *can solve low (as well as high) Mach number flow well*

our goal:

- for the *Euler equations with gravity* find a scheme that

    1. solves both low and high Mach numbers flow
    2. is well balanced
    3. is stable
    4. preserves kinetic energy for low Mach numbers

We begin by considering the ***homogeneous*** Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p = 0$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E \boldsymbol{v}) + \nabla \cdot (p \boldsymbol{v}) = 0$$

Goal: find a scheme that solves both low and high number well.

solve this using a finite volume scheme

astrophysicists like to use the Roe scheme

- Godunov-type numerical flux function (quasilinear, flux Jacobian $A(\boldsymbol{U}) \equiv \frac{\boldsymbol{F}}{\boldsymbol{U}}$)

$$\boldsymbol{F}_{i+1/2} = \frac{1}{2}\left[A^L \boldsymbol{U}^L_{i+1/2} + A^R \boldsymbol{U}^R_{i+1/2} - D\left(\boldsymbol{U}^R_{i+1/2} - \boldsymbol{U}^L_{i+1/2}\right)\right]$$

physical flux

"upwind term", numerical dissipation

"dissipation"

central flux

$$\frac{(F^k)_{i+\frac{1}{2}} - (F^k)_{i-\frac{1}{2}}}{\Delta x} = \frac{f^k(q_{i+1}) - f^k(q_{i-1})}{2\Delta x} + \frac{\left|\left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\mathrm{Roe}@i-1,i}\right| (q_i - q_{i-1}) - \left|\left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\mathrm{Roe}@i,i+1}\right| (q_{i+1} - q_i)}{2\Delta x}$$

$$\simeq \left.\frac{\partial f^k}{\partial q}\right._i + \left|\left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\mathrm{Roe}@i,i+1}\right| \Delta x (\partial_x^2 q)_i + \text{ higher oder terms}$$

$$\left.\frac{\partial f^k}{\partial q}\right._i \sim \begin{pmatrix} 0 & \mathcal{O}(1) & 0 \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(\frac{1}{M^2}) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}$$

$$\left|\left\langle \frac{\partial f^k}{\partial q} \right\rangle_{\mathrm{Roe}@i,i+1}\right| \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(M) & \mathcal{O}\left(\frac{1}{M}\right) \\ \mathcal{O}\left(\frac{1}{M}\right) & \mathcal{O}\left(\frac{1}{M}\right) & \mathcal{O}\left(\frac{1}{M}\right) \\ \mathcal{O}(M) & \mathcal{O}(M) & \mathcal{O}\left(\frac{1}{M}\right) \end{pmatrix}$$

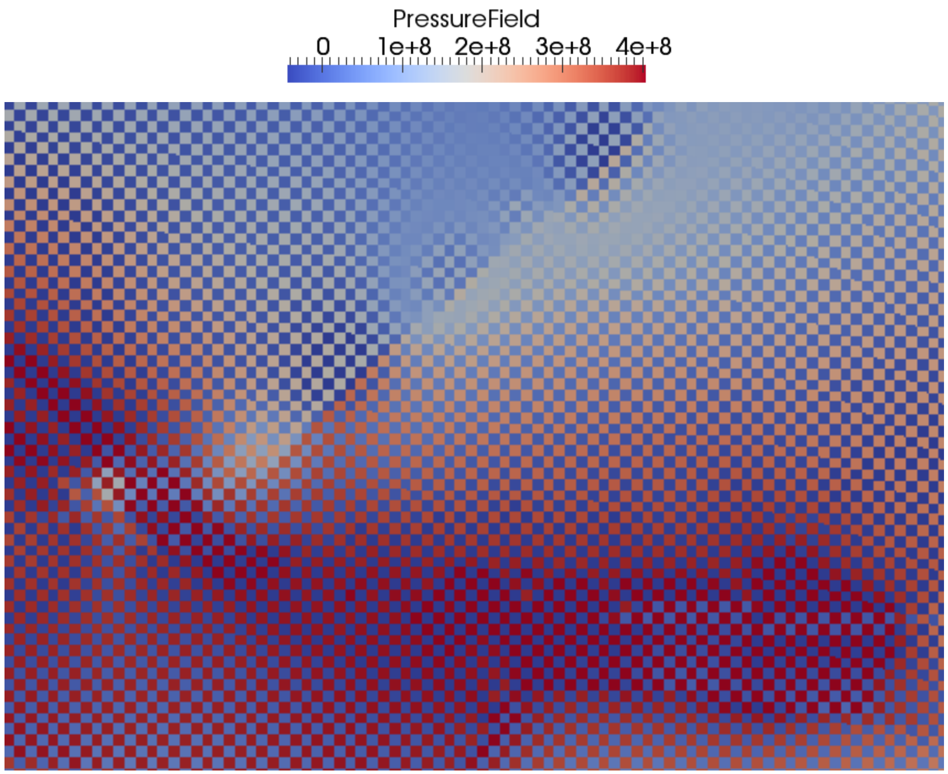
for low Mach number the dissipation *dominates* central flux thus one needs to ***modify*** the dissipation term

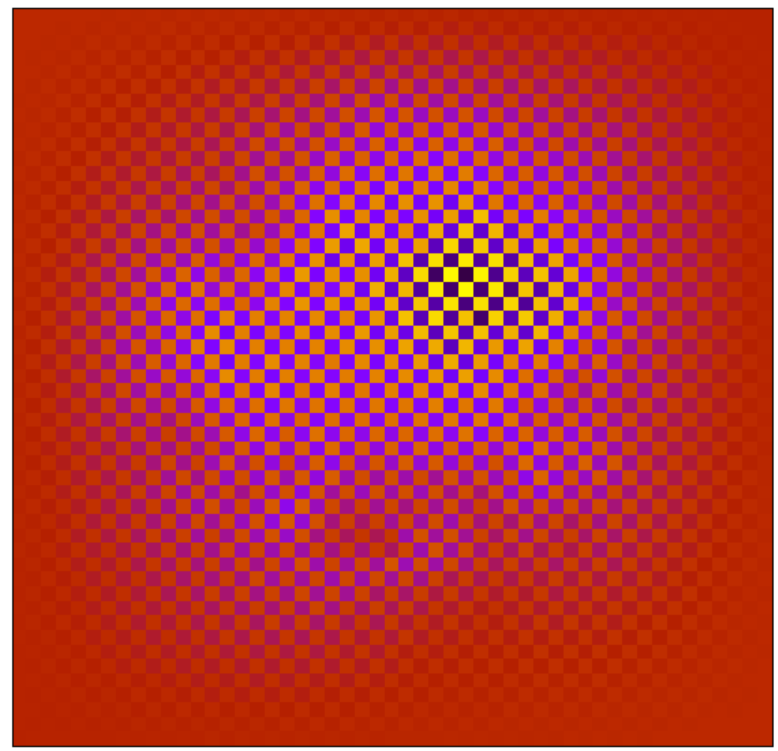
Find a modification such that

- for ***high*** Mach it behaves like the Roe scheme
- for ***low*** Mach number the dissipation of the stabilization term is sufficiently low
- for flow near the incompressible regime the total kinetic energy is well conserved
- the scheme is linearly stable

stability:

Near the incompressible regime, the solver discretizes the divergence free condition for the velocity, a checkerboard instability may arise.





We identified in a linear stability analysis that these modes survive neutral stability. Hence we do not allow that, and these instabilities go away.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{1}{2}\nu\left(A(\mathbf{U}_{i+1}^n - \mathbf{U}_{i-1}^n) - D(\mathbf{U}_{i+1}^n - 2\mathbf{U}_i^n + \mathbf{U}_{i-1}^n)\right)$$

$$\mathbf{U}_i^n = \sum_{k\in\mathbb{Z}} \mathbf{U}^n \exp(jik\Delta x)$$

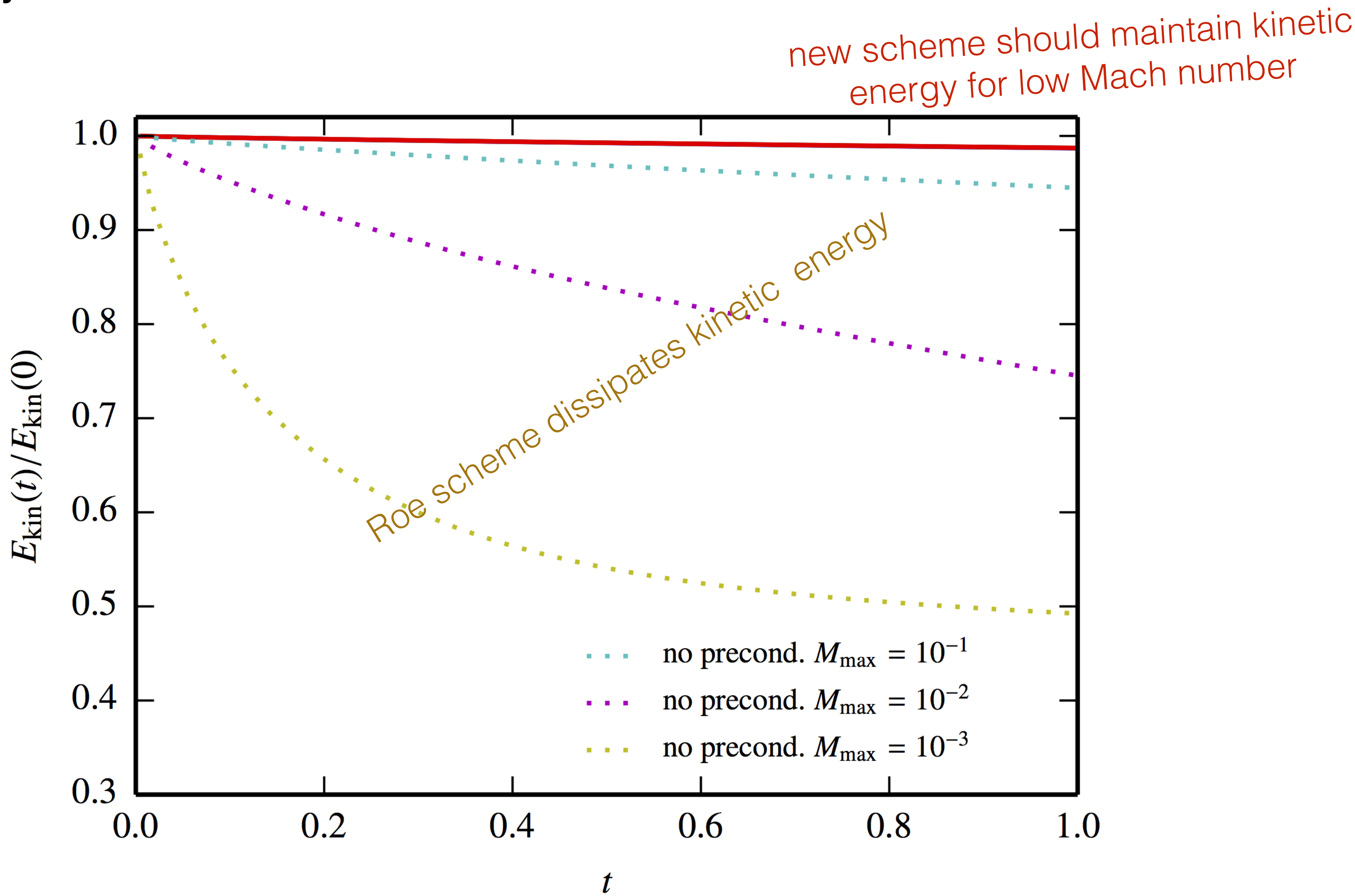
$$\mathbf{U}^{n+1} = \left[\mathrm{id} - \nu\left(Aj\sin\beta + D(1-\cos\beta)\right)\right]\mathbf{U}^n$$

$$\|\ \mathrm{id} - \nu\left(Aj\sin\beta + D(1-\cos\beta)\right)\| \overset{!}{<} 1$$

strict inequality

kinetic energy:

near the incompressible regime, one can show that the solution preserves kinetic energy.



It is tricky to find a dissipation matrix that satisfies all these conditions, especially in two space dimensions, but here is one example:

$$\begin{pmatrix} 1 & n_x \frac{\rho\delta M_\mathrm{r}}{c} & n_y \frac{\rho\delta M_\mathrm{r}}{c} & n_z \frac{\rho\delta M_\mathrm{r}}{c} & 0 \\ 0 & 1 & 0 & 0 & -n_x \frac{\delta}{\rho c M_\mathrm{r}} \\ 0 & 0 & 1 & 0 & -n_y \frac{\delta}{\rho c M_\mathrm{r}} \\ 0 & 0 & 0 & 1 & -n_z \frac{\delta}{\rho c M_\mathrm{r}} \\ 0 & n_x \rho c \delta M_\mathrm{r} & n_y \rho c \delta M_\mathrm{r} & n_z \rho c \delta M_\mathrm{r} & 1 \end{pmatrix}$$

$$\delta = \frac{1}{\mu} - 1 \qquad \mu = \min[1, \max(M_\mathrm{local})]$$

It adapts to the local Mach number, thus resorting back to Roe's scheme for Mach number ~ 1

consistency of the modified scheme with incompressible flow for M -> 0

e.g. pressure to highest order:

$$p(x,t) = p^{(0)}(x,t) + Mp^{(1)}(x,t) + M^2p^{(2)}(x,t) + \mathcal{O}(M^3)$$

to highest order:

$$0 = \partial_t \mathbf{U}_i + \frac{1}{2\Delta x}\left[\frac{1}{M^2}\begin{pmatrix} 0 \\ p_{i+1} - p_{i-1} \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{\gamma - 1}{M^2}\begin{pmatrix} 0 \\ p_{i+1} - 2p_i + p_{i-1} \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] + \mathcal{O}(M)$$

$$p_i^{(\ell)} - p_{i-1}^{(\ell)} = 0 \qquad \ell = 0, 1$$

Implicit time integration

- spatial discretization ("method of lines") yields

$$\frac{\partial \boldsymbol{U}}{\partial t} = \boldsymbol{R}(\boldsymbol{U})$$

- explicit time integration → stability requires limiting of time step due to sound speed:

$$\Delta t_{\mathrm{explicit}} \leq \mathrm{CFL}\frac{\Delta x}{|u+c|} \overset{u \ll c}{\approx} \mathrm{CFL}\frac{\Delta x}{c}$$

- implicit time integration → accuracy requires limiting of time step due to fluid velocity:

$$\Delta t_{\mathrm{implicit}} \leq \mathrm{CFL}\frac{\Delta x}{|u|}$$

- gain: implicit time step larger by factor $1/M$
- tests imply that implicit time stepping more efficient below $M \sim 0.2$

To solve this implicitly we need to solve a very large system of equations.

We can show that the condition number of our new scheme is rather good.

- for the *Euler equations* find a scheme that

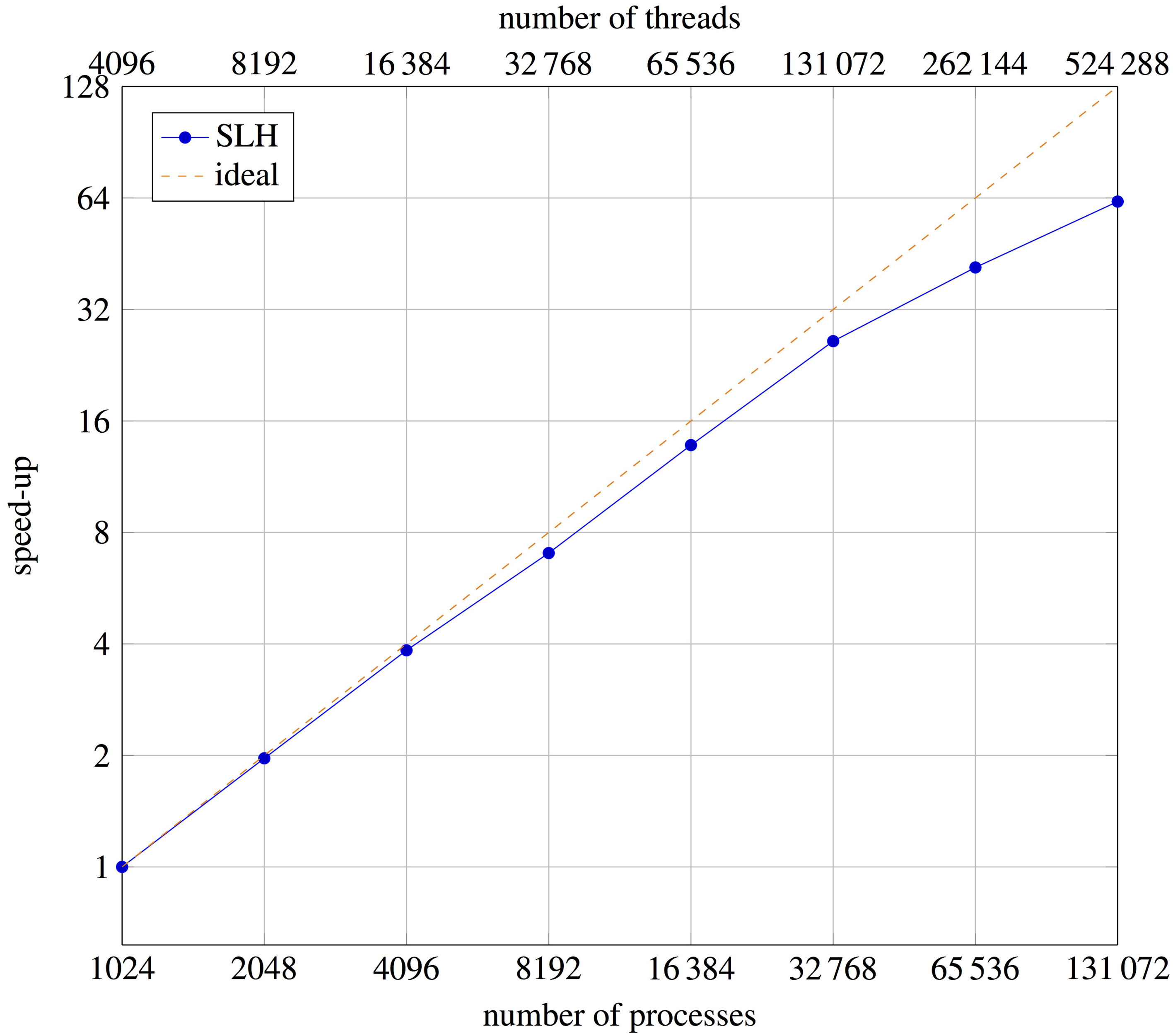
1. solves both low and high Mach numbers flow
2. is stable
3. preserves kinetic energy for low Mach numbers
4. the inversion of the large linear system arising from implicit time discretization has a good condition number

Scaling on HPC systems

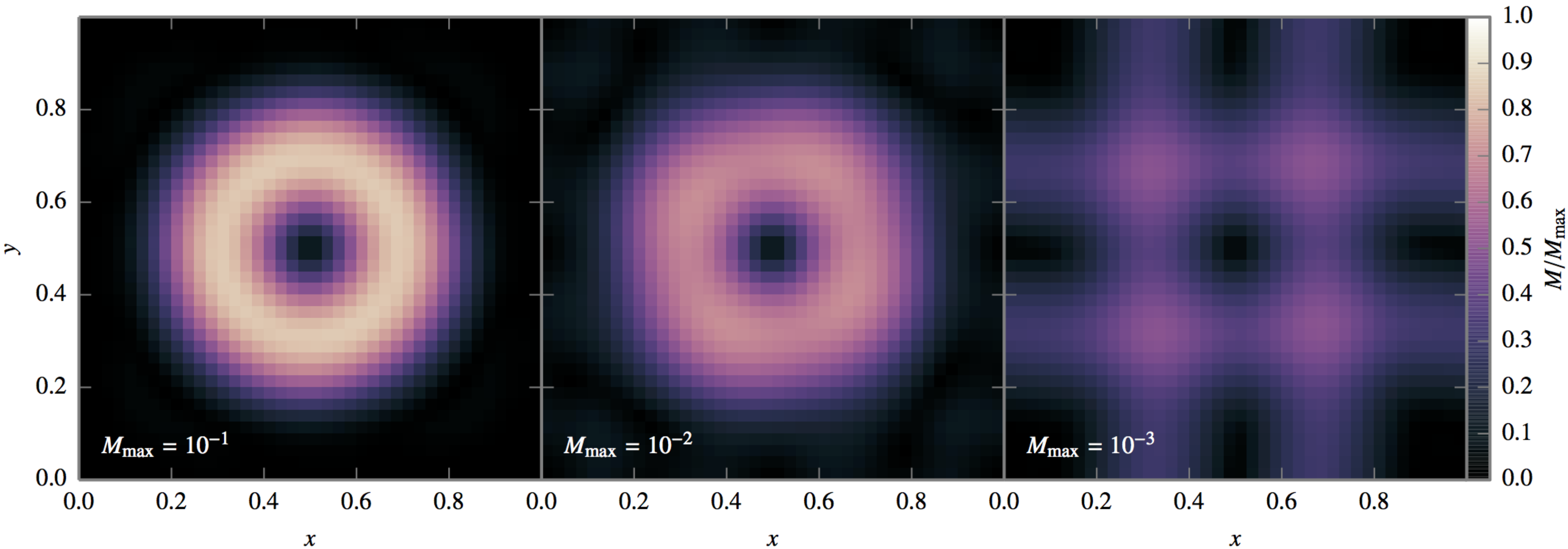
Strong scaling results on JUQUEEN

Test problem: Taylor–Green vortex on 512^3 grid

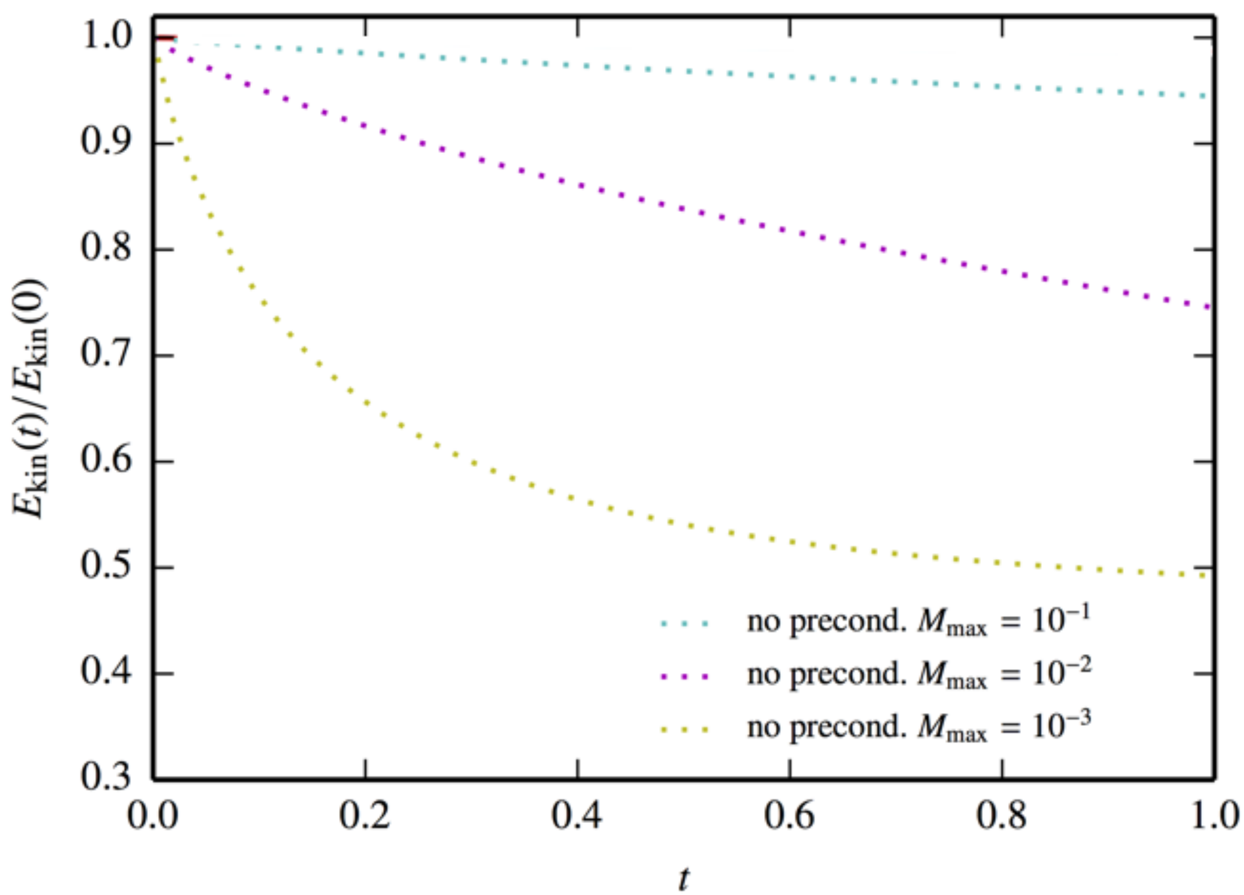
All runs performed with 16 MPI tasks per node and 4 threads per task using the BiCGSTAB(5) solver



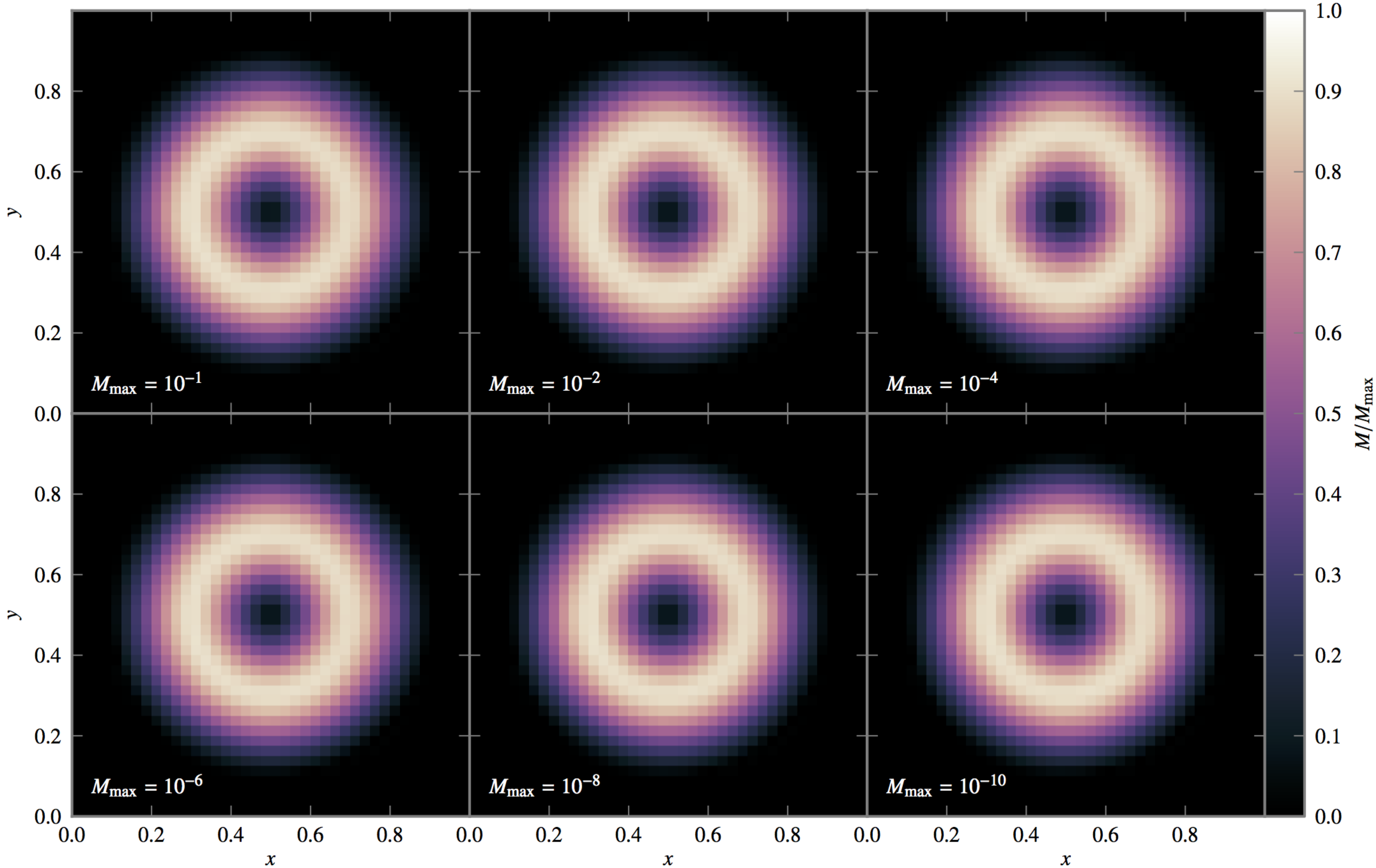
This is a “Gresho vortex” with an ***un-modified*** Roe scheme:



decay of kinetic energy for the ***un-modified*** Roe scheme:

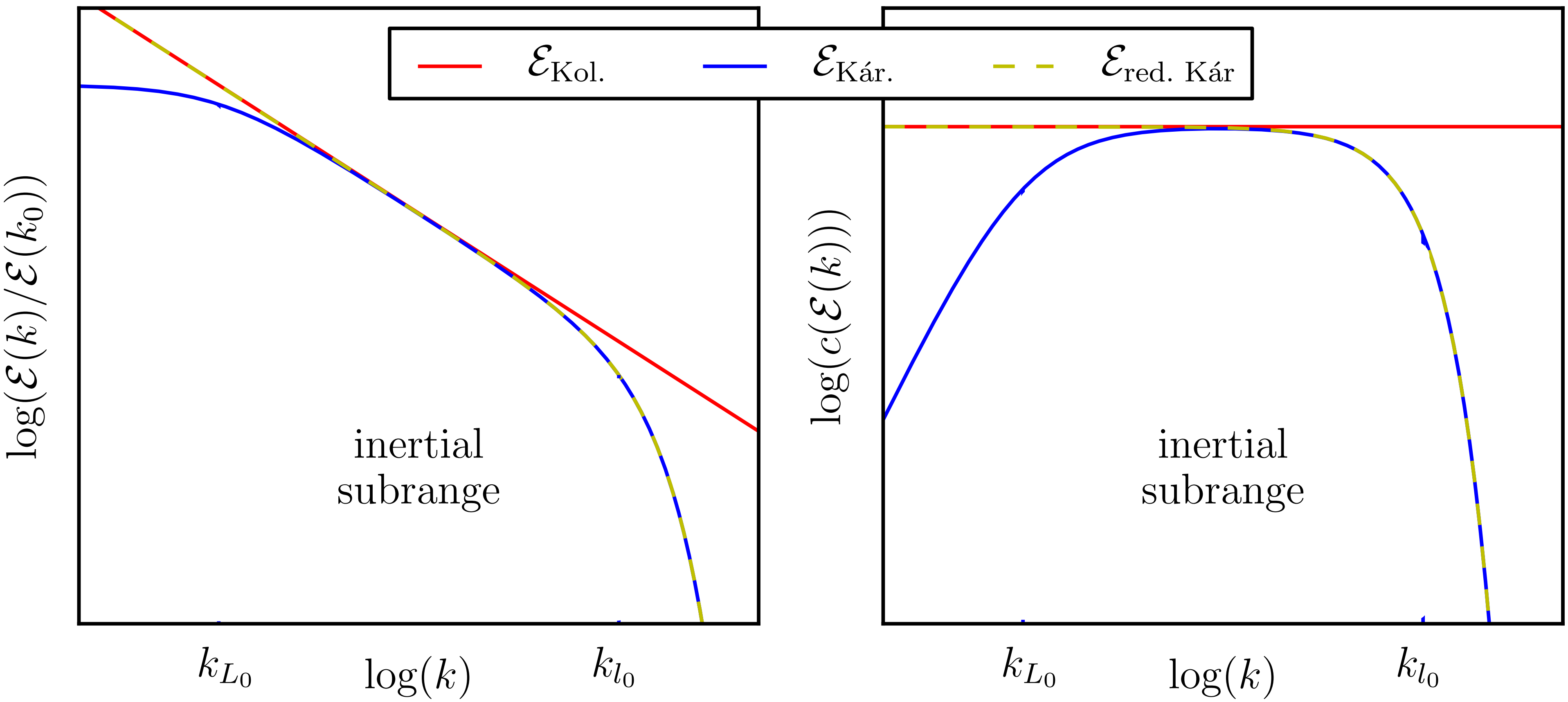


This is a “Gresho vortex” with the ***modified*** scheme tested down to $M_{\rm max} = 10^{-10}$



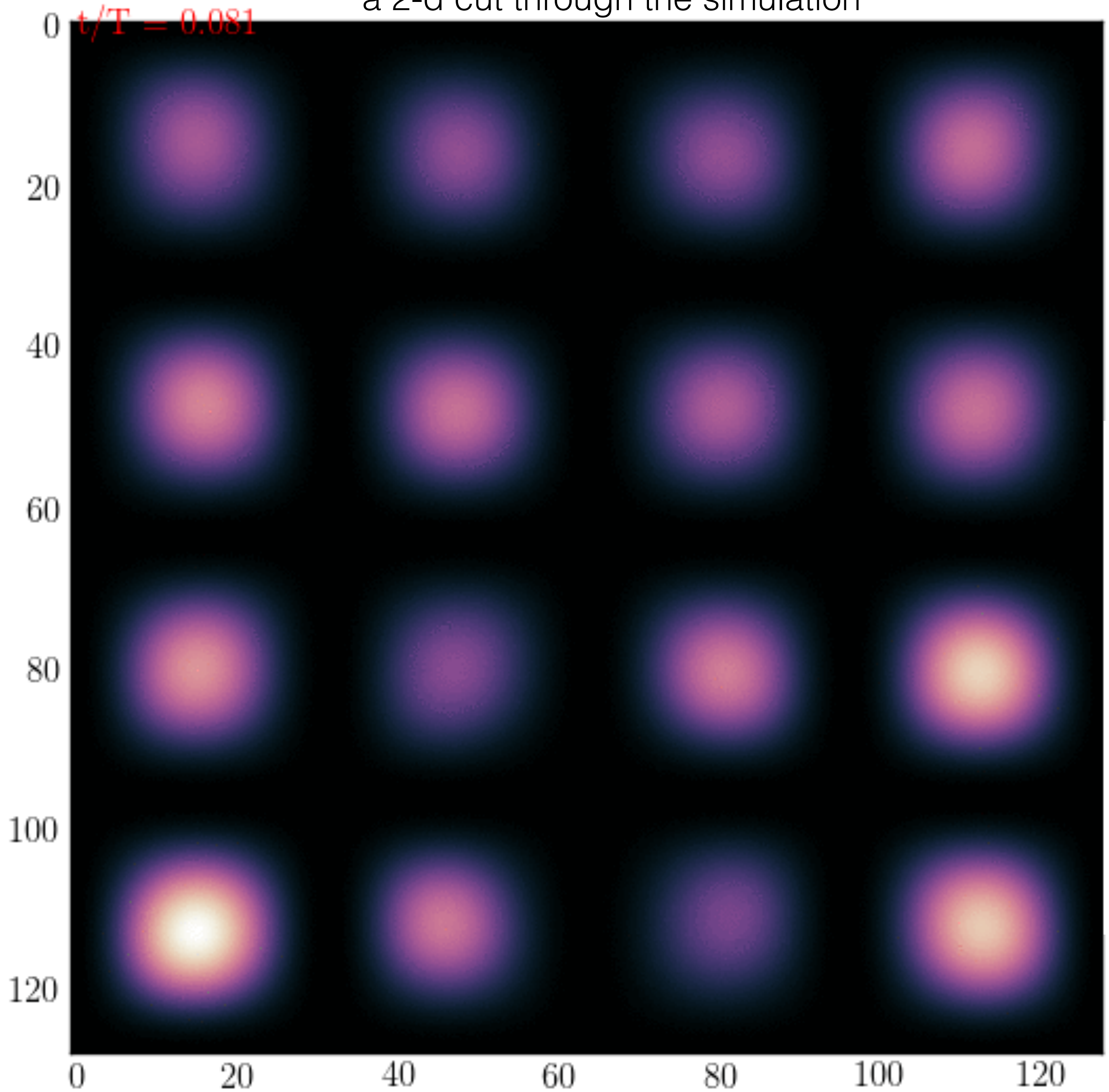
turbulence spectrum of incompressible flow à la Kolomogorov

same with presented as a compensated spectrum

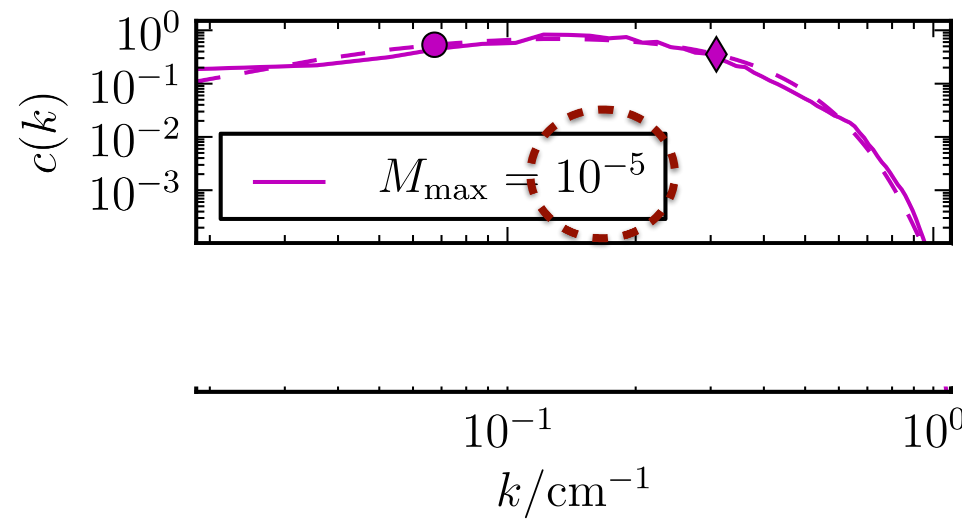


simulation of 3 space dim. decaying turbulence at very low Mach numbers

a 2-d cut through the simulation



compensated spectrum



We continue by considering the Euler equations ***with gravity***

$$
\begin{aligned}
&\rho_t + (\rho u)_x + (\rho v)_y = 0, \\
&(\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho u v)_y = -\rho \phi_x, \\
&(\rho v)_t + (\rho u v)_x + \left(\rho v^2 + p\right)_y = -\rho \phi_y, \\
&E_t + ((E+p)u)_x + ((E+p)v)_y = -\rho u \phi_x - \rho v \phi_y
\end{aligned}
$$

First we find a ***well balanced method***

The system of Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = -\rho \partial_x \phi \\ \partial_t E + \partial_x (u(E+p)) = -\rho u \partial_x \phi \end{cases}$$

- ρ: density
  u: velocity
  $E = \rho e + \rho u^2/2$: total energy, with e the internal energy
  $p = p(\rho, e)$: pressure given by a general law
  $\phi(x)$: gravitational potential (example: $\phi(x) = gx$)
- Hyperbolicity assumption:

$$c^2 := \partial_\rho p + \frac{p}{\rho^2} \partial_e p > 0$$

The system of Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p \right) = -\rho \partial_x \phi \\ \partial_t E + \partial_x (u(E+p)) = -\rho u \partial_x \phi \end{cases}$$

- We define
  - the vector of conservative variables $w = (\rho, \rho u, E)^T$,
  - the flux function $f(w) = (\rho u, \rho u^2 + p, u(E+p))^T$,
  - the source term $s(w) = (0, -\rho, -\rho u)^T$,

  to rewrite the system into the compact form

$$\partial_t w + \partial_x f(w) = s(w) \partial_x \phi.$$

- The set of physical admissible states is

$$\Omega = \left\{ w \in \mathbb{R}^3, \quad \rho > 0, \quad E - \rho u^2/2 > 0 \right\}.$$

The system of Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = -\rho \partial_x \phi \\ \partial_t E + \partial_x (u(E+p)) = -\rho u \partial_x \phi \end{cases}$$

Steady states at rest

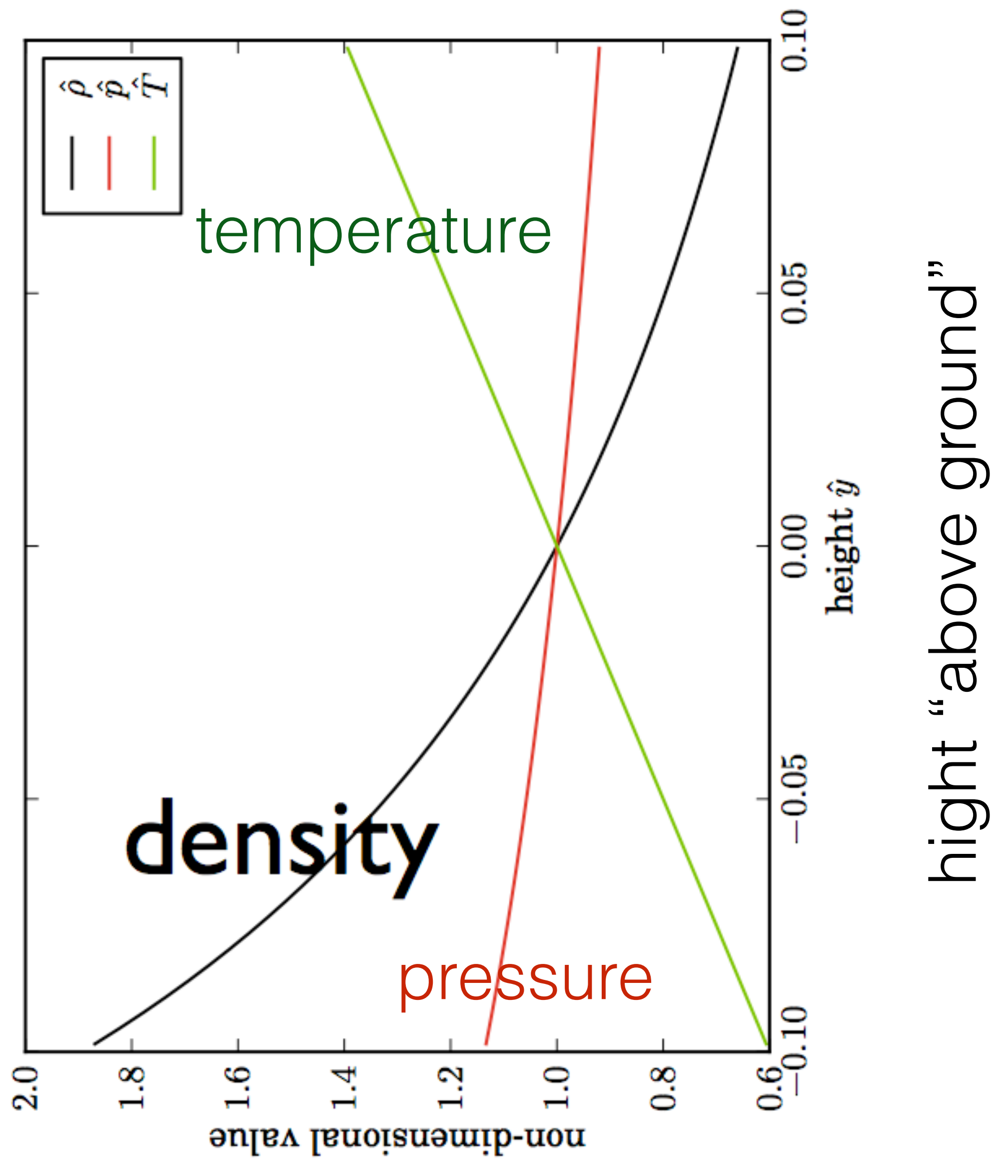
At the continuous level, the steady states at rest are governed by

$$\begin{cases} u \equiv 0, \\ \partial_x p = -\rho \partial_x \phi. \end{cases}$$

We cannot obtain an explicit expression of all the steady states.
$\rightarrow$ We have to define the steady states at the discrete level.

the steady states for Euler with gravity may be complicated

for any given
temperature profile
you can find a
hydrostatic
equilibrium



there are some well-balanced methods for Euler with gravity in the literature:

they maintain specific equilibria,

- Randy LeVeque
- Chi-Wang Shu
- Roger Käppeli, Sid Mishra
- Maria Lukacova

maintains all equilibria

- Alina Chertock

………
………

jointly with Praveen Chandrashekar:

"Higher oder entropy stable well-balanced schemes for Euler equations with gravity",

SIAM J. Sci. Computing (2015)

maintains specific equilibria: ideal gas

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) = -\rho \frac{\partial \phi}{\partial x}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(E + p)u = -\rho u \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = - \begin{bmatrix} 0 \\ \rho \\ \rho u \end{bmatrix} \frac{\partial \phi}{\partial x}$$

$$-\rho(x)\frac{\partial \phi}{\partial x} = p(x)\exp(-\psi(x))\frac{\partial}{\partial x}\exp(\psi(x)) \qquad \frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\int_{x_0}^{x}\frac{\phi'(s)}{RT(s)}\mathrm{d}s = -\frac{\phi'(x)}{RT(x)}$$

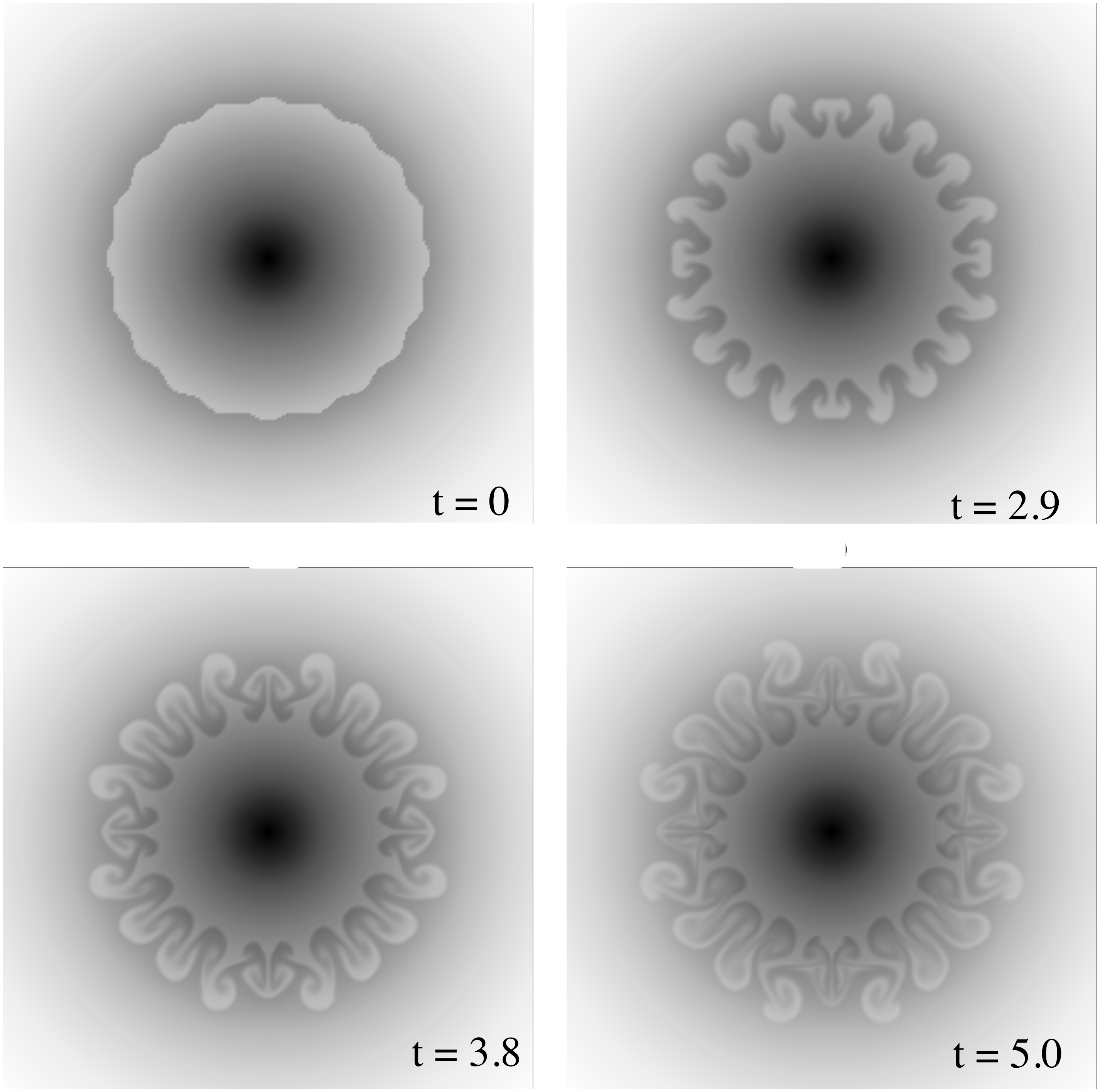
$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix}\exp(-\psi(x))\frac{\partial}{\partial x}\exp(\psi(x))$$

$$\frac{\mathrm{d}\boldsymbol{q}_i}{\mathrm{d}t} + \frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2}} - \hat{\boldsymbol{f}}_{i-\frac{1}{2}}}{\Delta x} = \mathrm{e}^{-\psi_i}\left(\frac{\mathrm{e}^{\psi_{i+\frac{1}{2}}} - \mathrm{e}^{\psi_{i-\frac{1}{2}}}}{\Delta x}\right)\begin{bmatrix} 0 \\ p_i \\ p_i u_i \end{bmatrix}$$

Well-balancing means preserving for the numerical scheme the following:

$$u_i = 0, \qquad p_i\exp(-\psi_i) = \mathrm{const}, \qquad \forall\ i \quad .$$

We show that with fluxes like Roe or HLLC this is can be achieved.



Rayleigh-Taylor instability in radial gravitational field using a well-balanced method

darker color indicates larger values

Christophe Berthon, Vivien Desveaux, Christian Klingenberg, Markus Zenk:
A well-balanced scheme for the Euler equation with a gravitational potential
Proceedings of the 7th International Symposium on Finite Volumes for Complex Applications (2014)

Christophe Berthon, Vivien Desveaux, Christian Klingenberg, Markus Zenk:
A well-balanced scheme to capture non-explicit steady states. Part II: Euler with gravity
International Journal for Numerical Methods in Fluids (2015)

this method maintains ***all*** equilibria

Discrete steady states and well-balanced scheme

- Space discretization: cells $[x_{i-1/2}, x_{i+1/2})$ with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$
- w_i^n: approximation of the solution of the system at time t^n on the cell $[x_{i-1/2}, x_{i+1/2})$
- Discretization of the potential ϕ: $\displaystyle \phi_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x)\,dx$

Definition (Discrete steady states)

An approximation $(w_i^n)_{i\in\mathbb{Z}}$ is a discrete steady state, if for all $i \in \mathbb{Z}$, we have

$$u_i^n = 0, \quad \text{and} \quad p_{i+1}^n - p_i^n = -\tfrac{\rho_i^n+\rho_{i+1}^n}{2}(\phi_{i+1} - \phi_i).$$

Definition (Well-balanced scheme)

A numerical scheme is well-balanced if for all discrete steady state $(w_i^n)_{i\in\mathbb{Z}}$, the scheme satisfies $w_i^{n+1} = w_i^n$, for all $i \in \mathbb{Z}$.

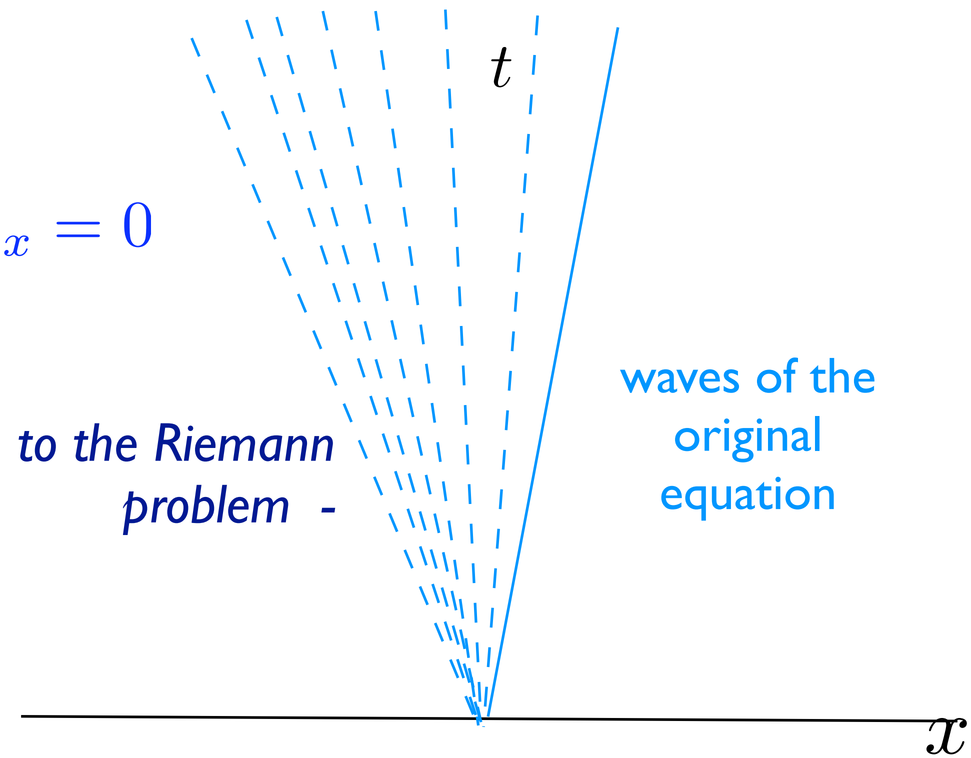
illustrate the idea of relaxation solver for a scalar conservation law

Jin - Xin relaxation:

replace: $$u_t + f(u)_x = 0$$

by:
$$\begin{aligned} u_t + \quad v_x &= 0 \\ v_t + a^2 u_x &= \frac{1}{\epsilon}(f(u) - v) \end{aligned}$$

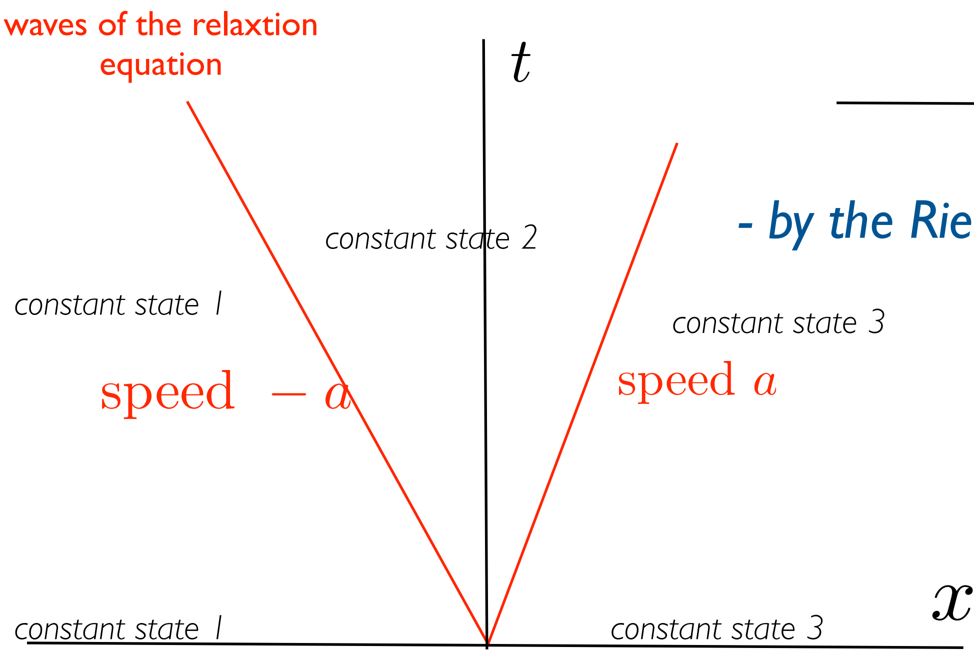
Goal:
find approximation to
Riemann problem of $u_t + f(u)_x = 0$



waves of the original equation

approximate the exact solution to the Riemann problem -

- by the Riemann solution of the relaxation system

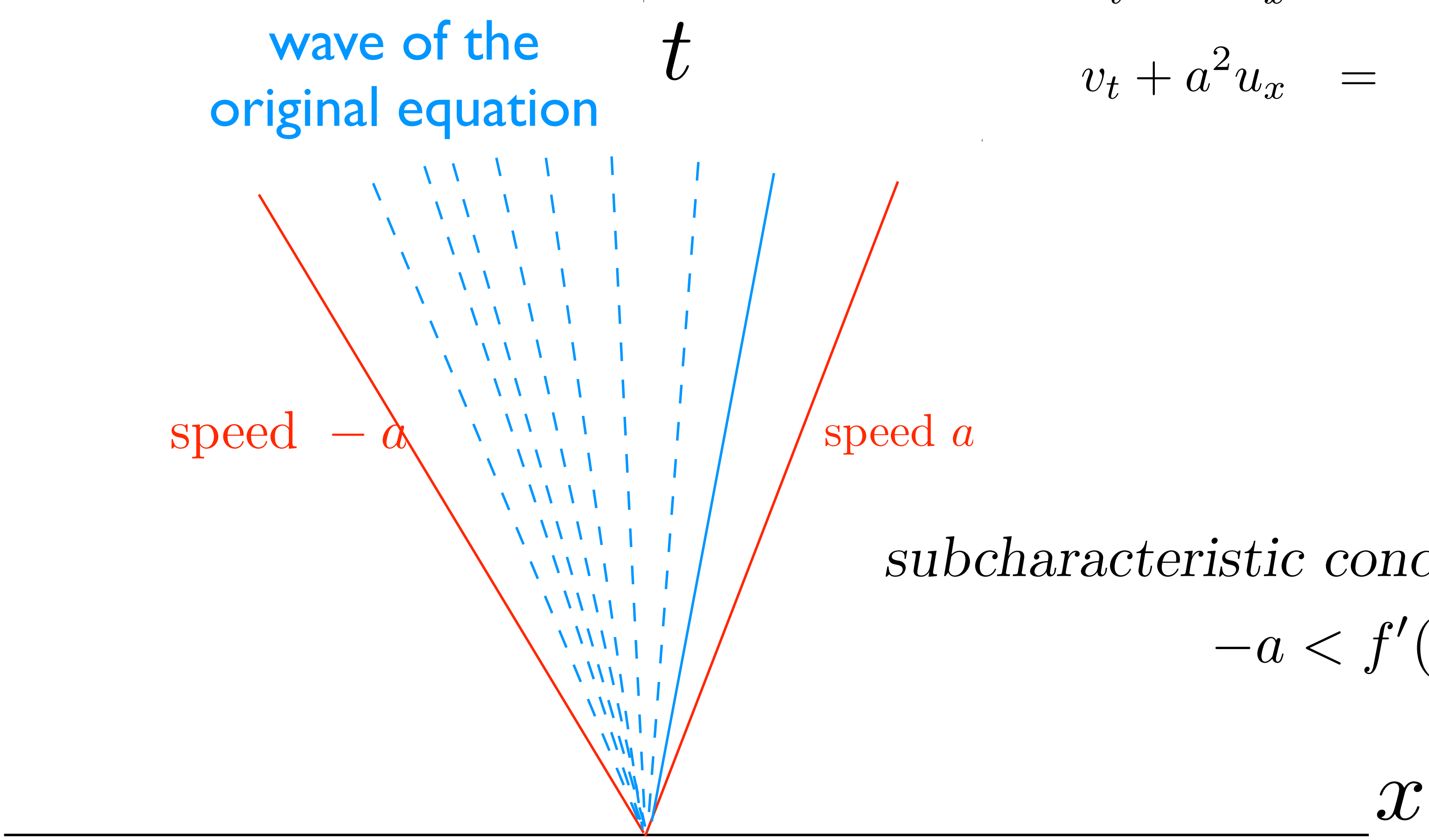


need to determine the speeds for the waves (an equality) -
satisfying the inequality of the subcharacteristic condition

the role of a^2

consider Riemann problem for the two sets of equations: $u_t + f(u)_x = 0$

$$\begin{aligned} u_t + \quad v_x &= 0 \\ v_t + a^2 u_x &= \frac{1}{\epsilon}(f(u) - v)) \end{aligned}$$



subcharacteristic condition:

$$-a < f'(u) < a$$

The Riemann solution to the relaxation system is easy to find because it is linearly degenerate.

In addition this approximate Riemann solver satisfies a discrete version of the entropy condition:

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{\Delta x_i}(G_{i+1/2} - G_{i-1/2}) \leq 0$$

entropy η
numerical entropy flux function $G(U_l, U_r)$

It is possible to determine the speeds of the approximate Riemann solver such that it is quite accurate while still maintaining the subcharacteristic condition which implies entropy consistency.

Summary for approximate Riemann solver via relaxation:

- with pencil and paper determine the stability condition
- for coding determine an algebraic formula for "optimal" speeds
- this will guarantee stability ("positivity")

In this spirit we embed system of compressible gas dynamics into a more "complete model".

For smooth solutions of the Euler equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x &= 0 \\ E_t + \big(u(E+p)\big)_x &= 0 \end{aligned}$$

we can write an evolution equation for the pressure:

$$(\rho p)_t + (\rho u p)_x + \rho^2 p'(\rho) u_x = 0$$

Replace p by a new dependant variable π and let c replace the soundspeed $\rho\sqrt{p'(\rho)}$

$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p-\pi}{\epsilon}$$

Siliciu (1990), Coquel, et.al. (1999)

the enlarged system has a small parameter $\epsilon > 0$ s.th.

$\epsilon > 0$ enlarged system

$\epsilon = 0$ original system

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + \pi)_x = 0$$

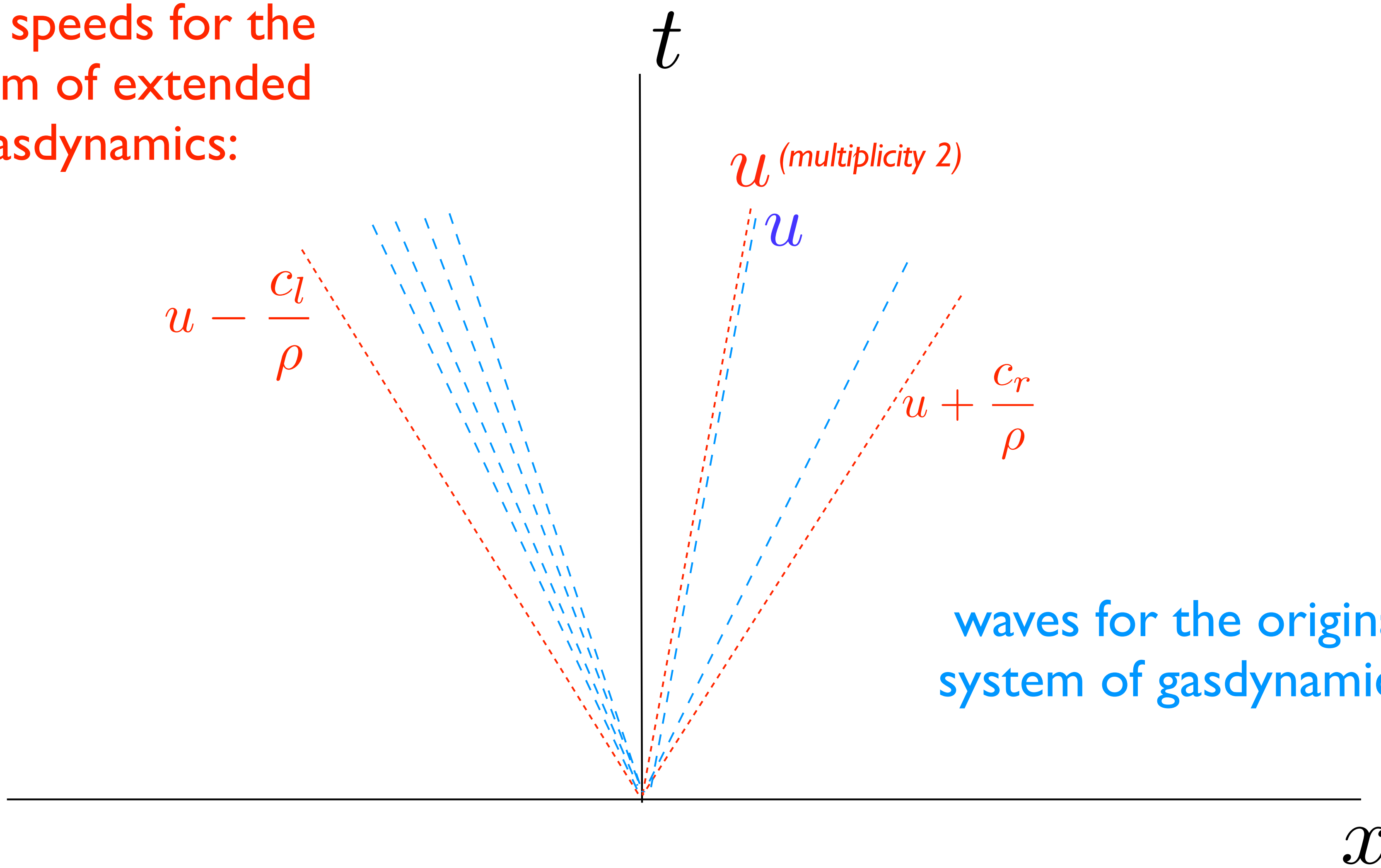
$$E_t + [(E + \pi)u]_x = 0$$

$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p - \pi}{\epsilon}$$

The constant c replaces the sound speed, which is a nonlinear function.

The advantage of the extended system is that by making the pressure a new dependent variable it easy to solve the Riemann problem for the homogeneous part of the extended system (all eigenvalues are degenerate).

wave speeds for the system of extended gasdynamics:



waves for the original system of gasdynamics:

Absolutely essential is the choice of the constant c (replacing the sound speed).

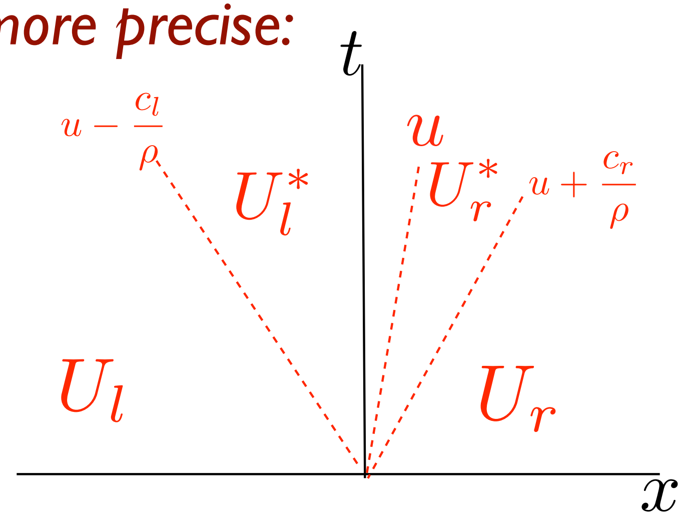
$$c > \rho\sqrt{p'(\rho)}$$ *“subcharacteristic condition”*

The choice of c determines the “stability’ of this relaxation.

It ensures an entropy inequality.

This is analyzed à la Chen, Levermore, Liu (1994) allowing for rigorous justification.

more precise:



$$\forall \rho \in [\rho_l, \rho_l^*], \quad \rho^2 \left(\frac{\partial p}{\partial \rho}\right)_s \leq c_l^2$$

$$\forall \rho \in [\rho_r, \rho_r^*], \quad \rho^2 \left(\frac{\partial p}{\partial \rho}\right)_s \leq c_r^2$$

For practical purposes, in order to devise a formula for a numerical scheme, one has to choose a particular value for c out of the possible values the inequality allows for.

$$
\text{if } p_r - p_l \geq 0, \quad \begin{cases} \dfrac{c_l}{\rho_l} = \sqrt{p'(\rho_l)} + \alpha \left(\dfrac{p_r - p_l}{\rho_r \sqrt{p'(\rho_r)}} + u_l - u_r \right)_+, \\[2ex] \dfrac{c_r}{\rho_r} = \sqrt{p'(\rho_r)} + \alpha \left(\dfrac{p_l - p_r}{c_l} + u_l - u_r \right)_+, \end{cases}
$$

$$
\text{if } p_r - p_l \leq 0, \quad \begin{cases} \dfrac{c_r}{\rho_r} = \sqrt{p'(\rho_r)} + \alpha \left(\dfrac{p_l - p_r}{\rho_l \sqrt{p'(\rho_l)}} + u_l - u_r \right)_+, \\[2ex] \dfrac{c_l}{\rho_l} = \sqrt{p'(\rho_l)} + \alpha \left(\dfrac{p_r - p_l}{c_r} + u_l - u_r \right)_+. \end{cases}
$$

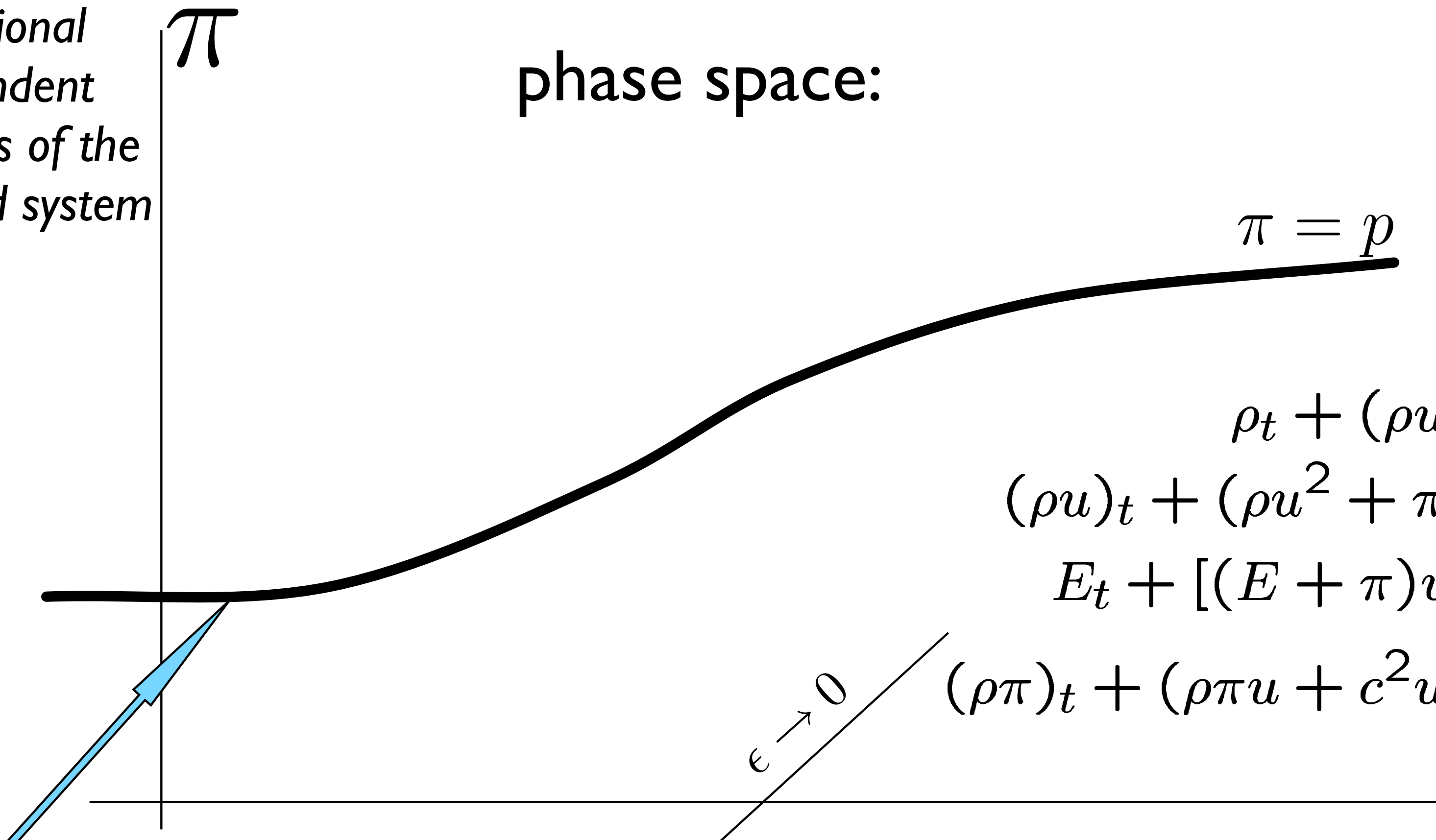
Bouchut (2004)

This ensures the optimal properties of this approximate Riemann solver.

Illustrate relaxation solver in phase space

phase space:

additional dependent variables of the extended system



equilibrium manifold

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + \pi)_x = 0$$
$$E_t + [(E + \pi)u]_x = 0$$
$$(\rho\pi)_t + (\rho\pi u + c^2 u)_x = \rho\frac{p - \pi}{\epsilon}$$

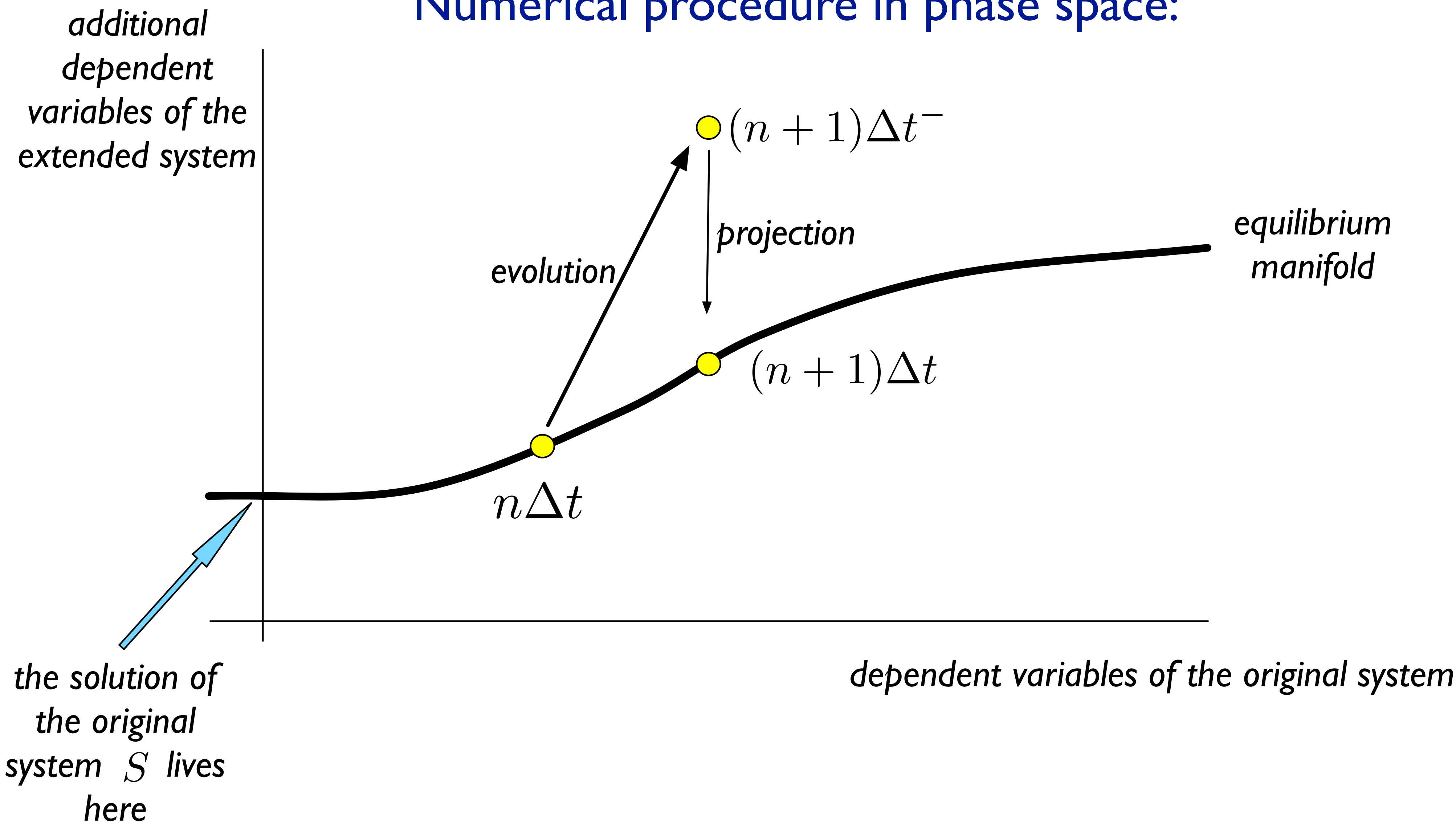
the solution of the original system lives here

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + \big(u(E + p)\big)_x = 0$$

dependent variables of the original system

ρ, u, E

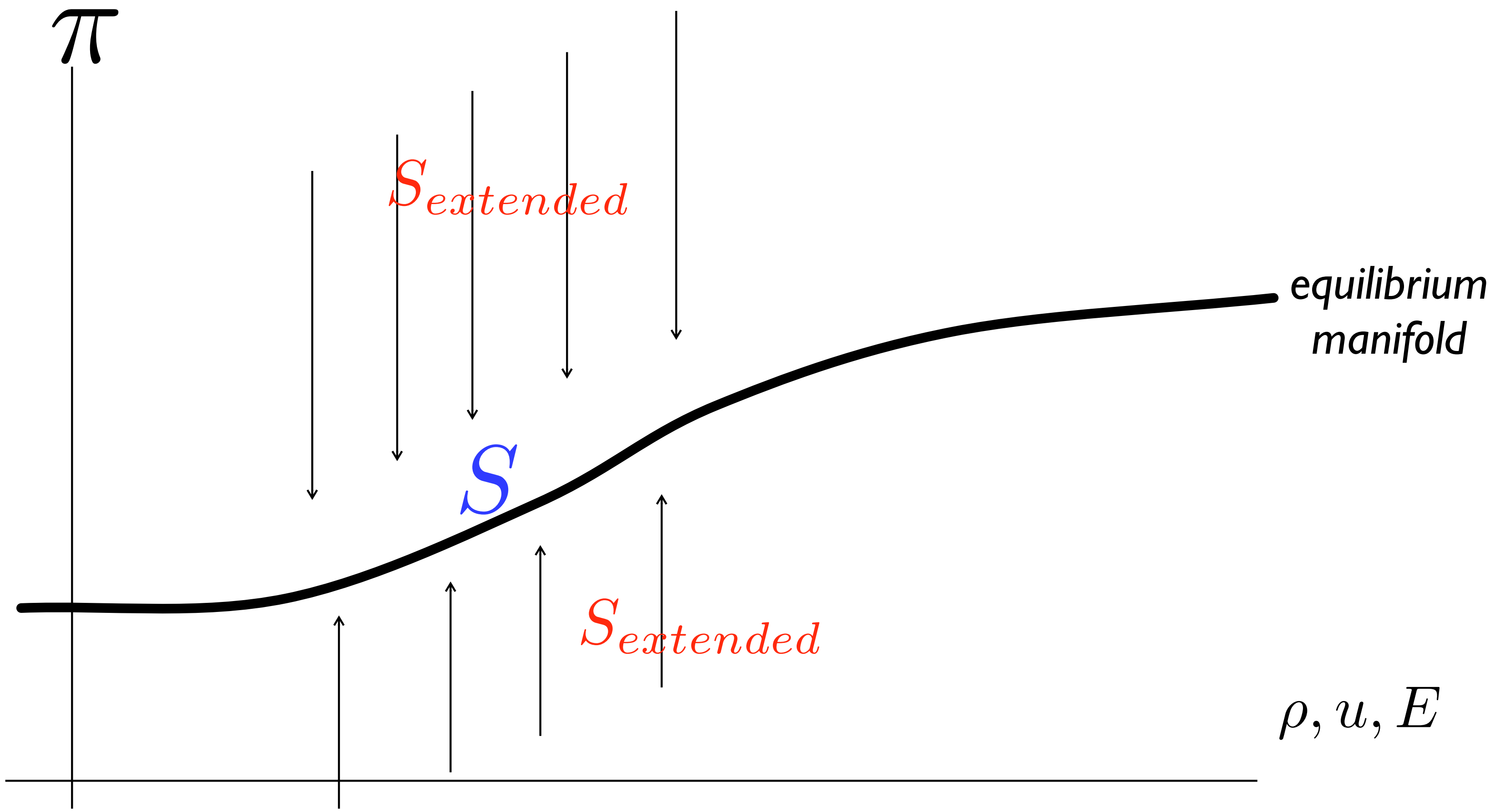
Numerical procedure in phase space:



This results in a numerical method for the original system.

It is possible to extend the entropy S of the original system of gas dynamics to an entropy $S_{extended}$ of the system of extended gas dynamics

such that for $\epsilon \longrightarrow 0$ the extended entropy converges to the original entropy.



this procedure translates Riemann solvers for the extended system to Riemann solvers for the original system

- preserves $\rho \geq 0$
- can handle vacuum
- this ensures that the "second law of thermodynamics" is satisfied by the numerical solution of our original system

next we show how to find a relaxation method for Euler with gravity based on this relaxation idea

The Suliciu model for the Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho \partial_x \phi \\ \partial_t E + \partial_x(u(E + \pi)) = -\rho u \partial_x \phi \\ \partial_t \rho\pi + \partial_x(u(\rho\pi + \nu^2)) = \frac{\rho}{\varepsilon}(p(\rho, e) - \pi) \end{cases}$$

we shall see, this particular relaxation does not work

The relaxation parameter $\nu > 0$ must satisfy the Whitham condition:
or subcharacteristic cond.:

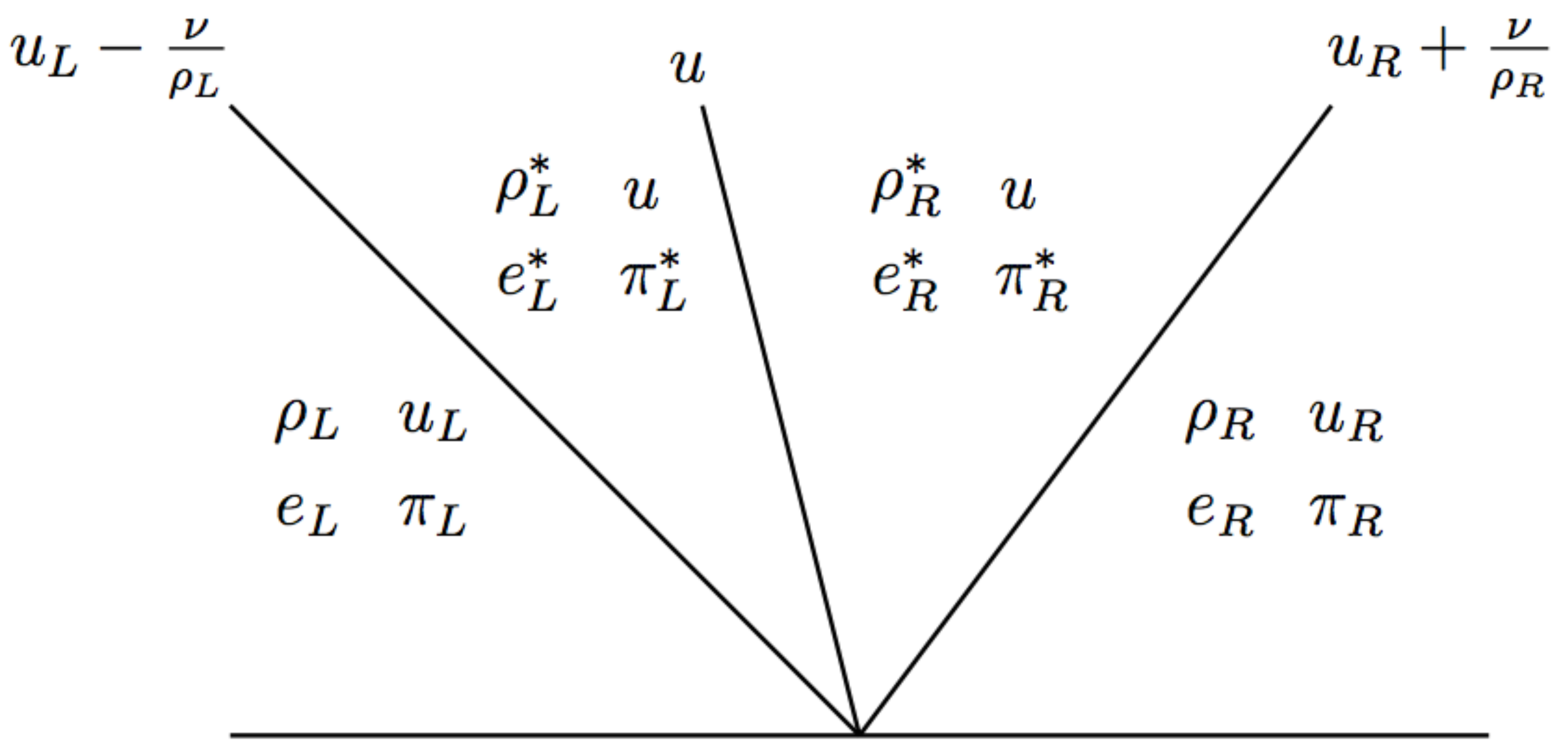
$$\nu^2 > \rho^2 c^2.$$

Eigenvalues	Riemann invariants
$u \pm \frac{\nu}{\rho}$	$u \pm \frac{\nu}{\rho}, \quad \pi \mp \nu u, \quad \nu^2 e - \frac{\pi^2}{2}, \quad \phi$
$u\ (\times 2)$	$u, \quad \pi, \quad \phi$
0	$\rho u, \quad \pi + \frac{\nu^2}{\rho}, \quad \nu^2 e - \frac{\pi^2}{2}, \quad \phi + \frac{u^2}{2} - \frac{\nu^2}{2\rho^2}$

Difficulties to compute the solution of the Riemann problem:

- the order of the eigenvalues is not determined *a priori*
- there are strong nonlinearities in the Riemann invariants for the eigenvalue 0

The Riemann problem



NEXT: new idea: artificially change the speed of u

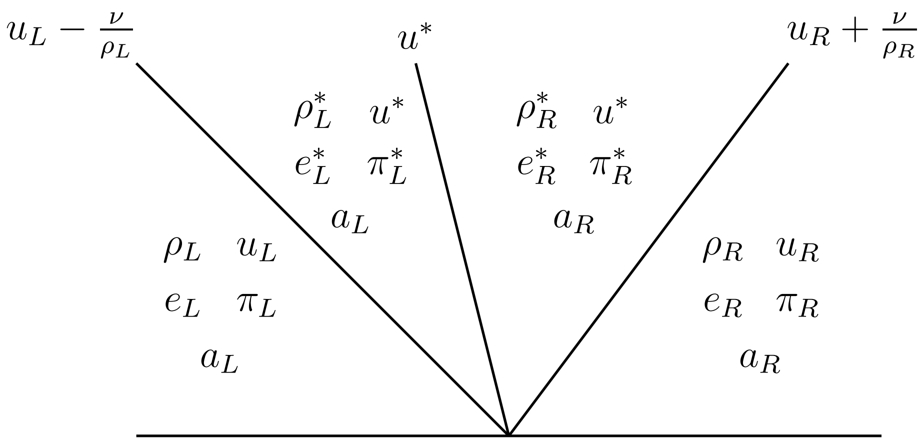
Relaxation model with moving gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho \partial_x a \\ \partial_t E + \partial_x(u(E+\pi)) = -\rho u \partial_x a \\ \partial_t \rho\pi + \partial_x(u(\rho\pi + \nu^2)) = \frac{\rho}{\varepsilon}(p(\rho, e) - \pi) \\ \partial_t a + u\partial_x a = \frac{1}{\varepsilon}(\phi - a) \end{cases}$$

Eigenvalues	Riemann invariants
$u \pm \frac{\nu}{\rho}$	$u \pm \frac{\nu}{\rho}, \quad \pi \mp \nu u, \quad \nu^2 e - \frac{\pi^2}{2}, \quad a$
$u \ (\times 3)$	u

- The order of the eigenvalues is fixed: $u - \frac{\nu}{\rho} < u < u + \frac{\nu}{\rho}$
- There is a missing invariant for the eigenvalue u
  $\Rightarrow$ we need a closure equation

The Riemann problem



- 7 unknowns: $u^*, \ \rho^*_{L,R}, \ e^*_{L,R}, \ \pi^*_{L,R}$
- 6 equations given by the Riemann invariants $u \pm \frac{\nu}{\rho}, \ \pi \mp \nu u, \ \nu^2 e - \frac{\pi^2}{2}$
- Closure equation:

missing

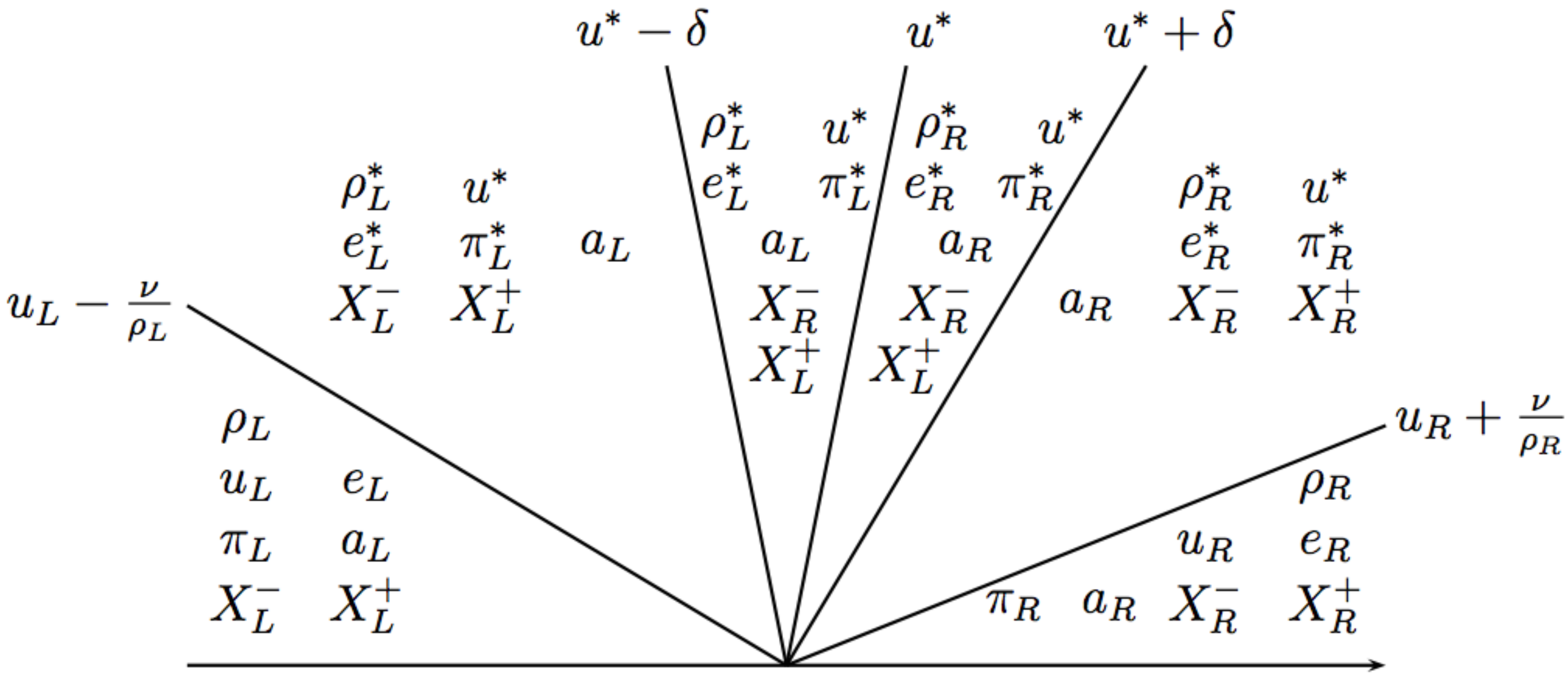
Solution of the Riemann problem

equations for u* missing

$$\pi_L^* = \pi_L + \nu(u_L - u^*) \qquad \pi_R^* = \pi_R + \nu(u^* - u_R)$$

$$\frac{1}{\rho_L^*} = \frac{1}{\rho_L} + \frac{u^* - u_L}{\nu} \qquad \frac{1}{\rho_R^*} = \frac{1}{\rho_R} + \frac{u_R - u^*}{\nu}$$

$$e_L^* = e_L + \frac{{\pi_L^*}^2 - {\pi_L}^2}{2\nu^2} \qquad e_R^* = e_R + \frac{{\pi_R^*}^2 - {\pi_R}^2}{2\nu^2}$$



next idea: we introduce two new speeds: u*+∂ and u*- ∂

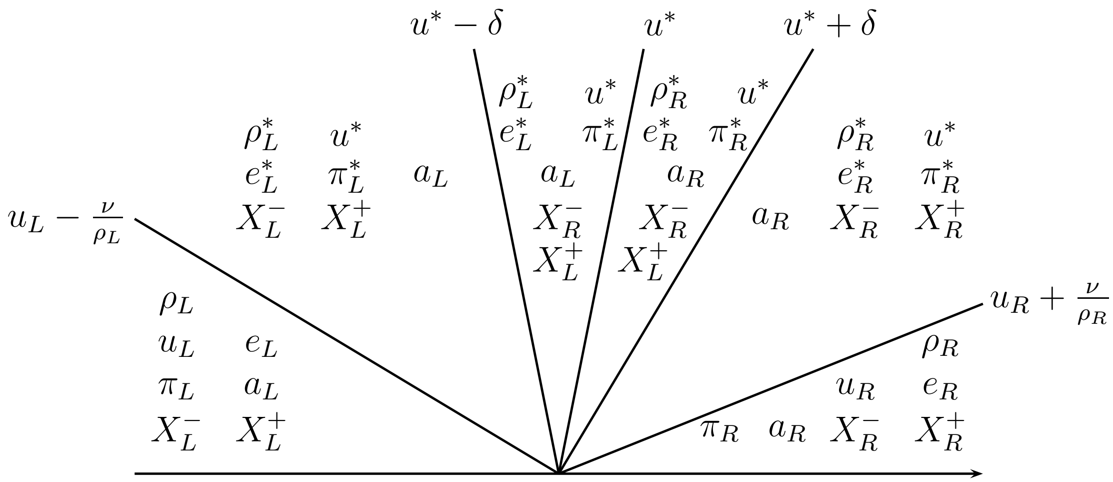
Reformulation into a fully determined model

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\frac{X^- + X^+}{2}\partial_x a \\ \partial_t E + \partial_x(u(E+\pi)) = -\frac{X^-+X^+}{2} u \partial_x a \\ \partial_t \rho\pi + \partial_x(u(\rho\pi + \nu^2)) = \frac{\rho}{\varepsilon}(p(\rho,e) - \pi) \\ \partial_t a + u\partial_x a = \frac{1}{\varepsilon}(\phi - a) \\ \partial_t X^- + (u-\delta)\partial_x X^- = \frac{1}{\varepsilon}(\rho - X^-) \\ \partial_t X^+ + (u+\delta)\partial_x X^+ = \frac{1}{\varepsilon}(\rho - X^+) \end{cases}$$

Eigenvalues	Riemann invariants
$u \pm \frac{\nu}{\rho}$	$u \pm \frac{\nu}{\rho}$, $\pi \mp \nu u$, $\nu^2 e - \frac{\pi^2}{2}$, a, X^-, X^+
u $(\times 3)$	u, $\pi + \frac{X^-+X^+}{2} a$, X^-, X^+
$u - \delta$	ρ, u, e, π, a, X^+
$u + \delta$	ρ, u, e, π, a, X^-

- For δ small enough, the order of the eigenvalues is fixed: $u - \frac{\nu}{\rho} < u - \delta < u < u + \delta < u + \frac{\nu}{\rho}$
- There is a full set of Riemann invariants

The Riemann problem for the reformulated model



- 7 unknowns:
  $u^*,\ \rho^*_{L,R},\ e^*_{L,R},\ \pi^*_{L,R}$
- 7 equations given by the Riemann invariants
  $u\pm\frac{\nu}{\rho},\ \pi\mp\nu u,\ \nu^2 e-\frac{\pi^2}{2},$
  $\pi+\frac{X^-+X^+}{2}a$

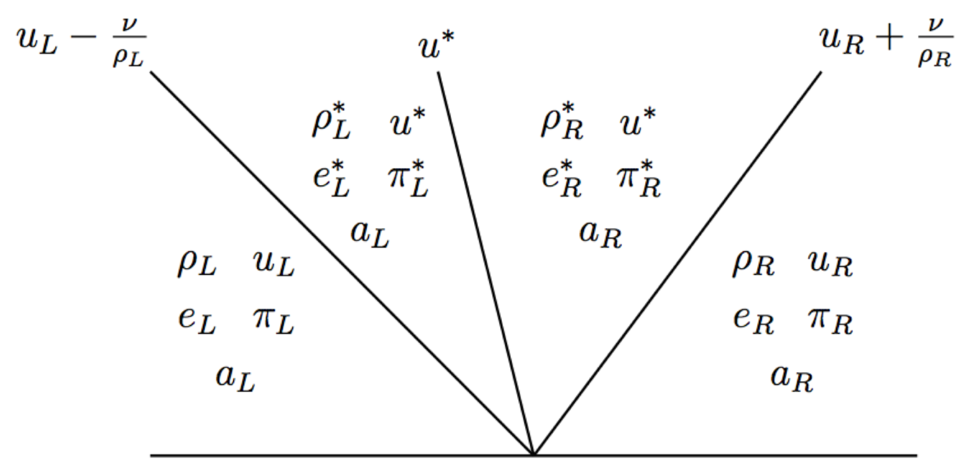
- The equations coming from the Riemann invariants $u\pm\frac{\nu}{\rho}$, $\pi\mp\nu u$ and $\nu^2 e-\frac{\pi^2}{2}$ are the same as in the previous model.
- The last equation is $\boxed{\pi_R^* - \pi_L^* = -\frac{X_R^- + X_L^+}{2}(a_R - a_L)}$. For an initial data at the relaxation equilibrium (i.e. $\pi = p(\rho, e)$, $a = \phi$, $X^\pm = \rho$), we recover the closure equation of the previous model.

$$\boxed{p_{i+1}^n - p_i^n = -\frac{\rho_i^n + \rho_{i+1}^n}{2}(\phi_{i+1} - \phi_i)}$$

this is our well-balanced condition !

another way of seeing this is to say:

we can solve the missing equations for the "*changing the speed u** " relaxation problem by using the well-balanced constraint.



*artificially change the speed of u**

The relaxation scheme

The relaxation scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F(w_i^n, w_{i+1}^n) - F(w_{i-1}^n, w_i^n)\right)$$
$$+ \frac{\Delta t}{2}\left(s^+(w_{i-1}^n, w_i^n)\frac{\phi_i - \phi_{i-1}}{\Delta x} + s^-(w_i^n, w_{i+1}^n)\frac{\phi_{i+1} - \phi_i}{\Delta x}\right),$$

where the numerical flux is defined by

$$f(w_L,w_R)=\begin{cases} \left(\rho_L u_L, \quad \rho_L u_L^2+p_L, \quad u_L(E_L+p_L)\right)^T & \text{if } u_L-\frac{\nu}{\rho_L}>0, \\ \left(\rho_L^* u^*, \quad \rho_L^*(u^*)^2+\pi_L^*, \quad u^*(E_L^*+\pi_L^*)\right)^T & \text{if } u_L-\frac{\nu}{\rho_L}<0<u^*, \\ \left(\rho_R^* u^*, \quad \rho_R^*(u^*)^2+\pi_R^*, \quad u^*(E_R^*+\pi_R^*)\right)^T & \text{if } u^*<0<u_R+\frac{\nu}{\rho_R}, \\ \left(\rho_R u_R, \quad \rho_R u_R^2+p_R, \quad u_R(E_R+p_R)\right)^T & \text{if } u_R+\frac{\nu}{\rho_R}<0, \end{cases}$$

and the numerical source terms are defined by

$$s^+(w_L,w_R)=-(\mathrm{sgn}(u^*)+1)\left(0, \frac{\rho_L+\rho_R}{2}, \frac{\rho_L+\rho_R}{2}u^*\right)^T,$$

$$s^-(w_L,w_R)=(\mathrm{sgn}(u^*)-1)\left(0, \frac{\rho_L+\rho_R}{2}, \frac{\rho_L+\rho_R}{2}u^*\right)^T.$$

Properties of the relaxation scheme

Theorem (Well-balancedness)

The relaxation scheme preserves the steady states at rest:

$$\forall i \in \mathbb{Z}, \begin{cases} u_i^n = 0 \\ p_{i+1}^n - p_i^n = -\frac{\rho_i^n + \rho_{i+1}^n}{2}(\phi_{i+1} - \phi_i) \end{cases} \Rightarrow \forall i \in \mathbb{Z},\ w_i^{n+1} = w_i^n$$

Theorem (Robustness)

Assume the parameter ν satisfies the following inequalities:

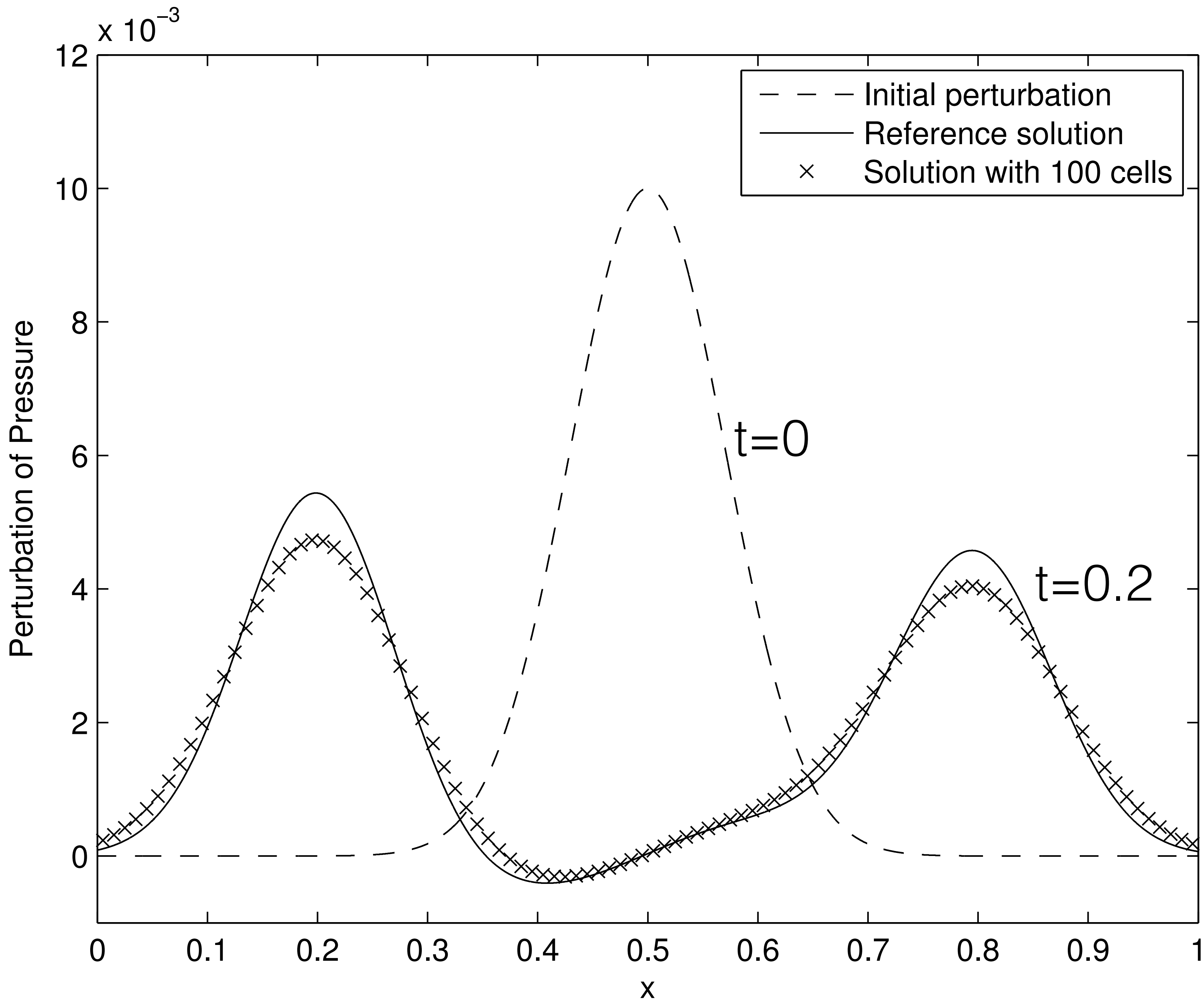
$$u_L - \frac{\nu}{\rho_L} < u^* < u_R + \frac{\nu}{\rho_R}, \quad e_L + \frac{{\pi_L^*}^2 - {p_L}^2}{2\nu^2} > 0, \quad e_R + \frac{{\pi_R^*}^2 - {p_R}^2}{2\nu^2} > 0.$$

Assume the following CFL condition is satisfied:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} |u_i^n \pm \nu / \rho_i^n| \leq \frac{1}{2}$$

Then the relaxation scheme preserves the set of physical states:

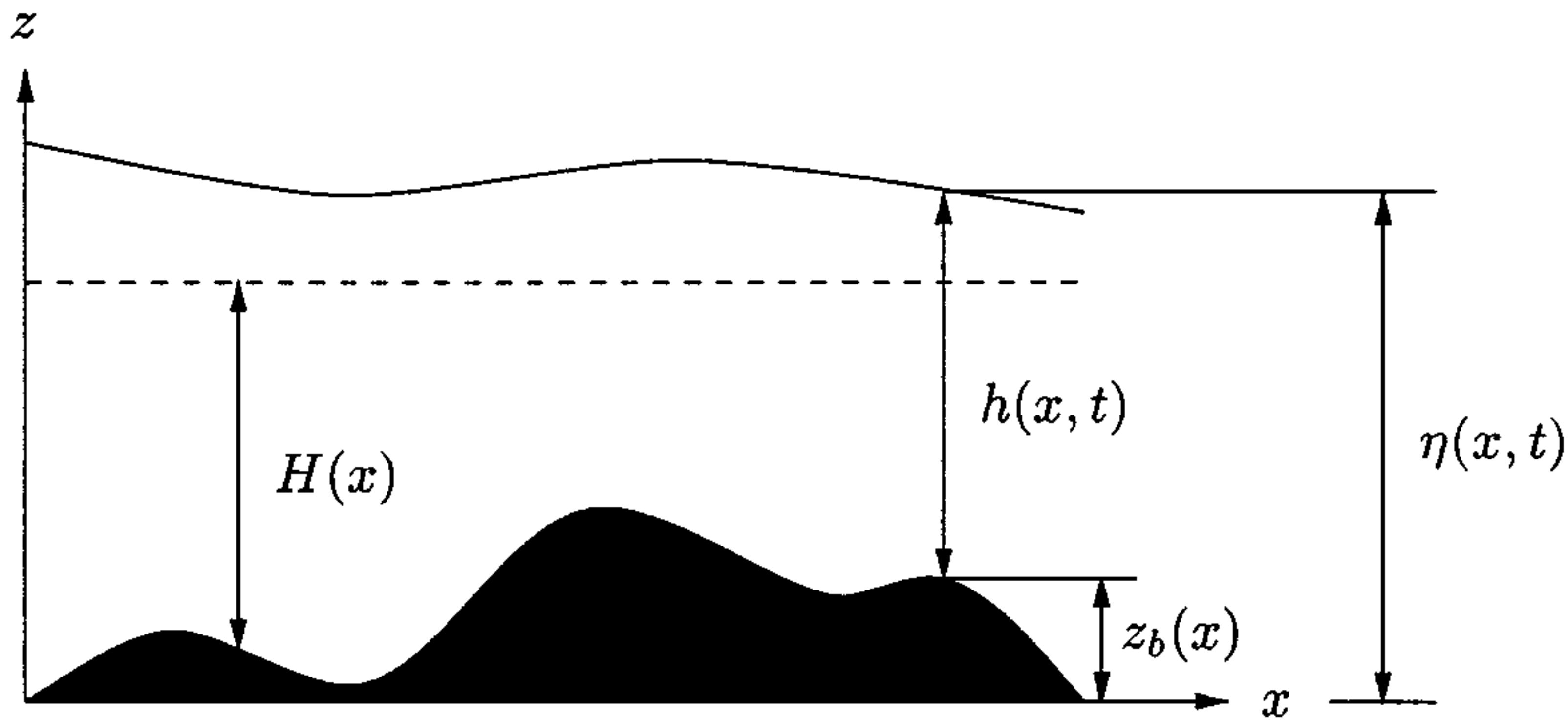
$$\forall i \in \mathbb{Z},\ \rho_i^n > 0 \text{ and } e_i^n > 0 \quad \Rightarrow \quad \forall i \in \mathbb{Z},\ \rho_i^{n+1} > 0 \text{ and } e_i^n > 0.$$



Perturbation of an equilibrium state; profile of the pressure disturbance at time t = 0.2

This is a first order well-balanced method. This can be made higher order by using a variant of “surface gradient method” from shallow water equations:

$$\text{use } \eta \text{ instead of } h \text{ for reconstruction}$$

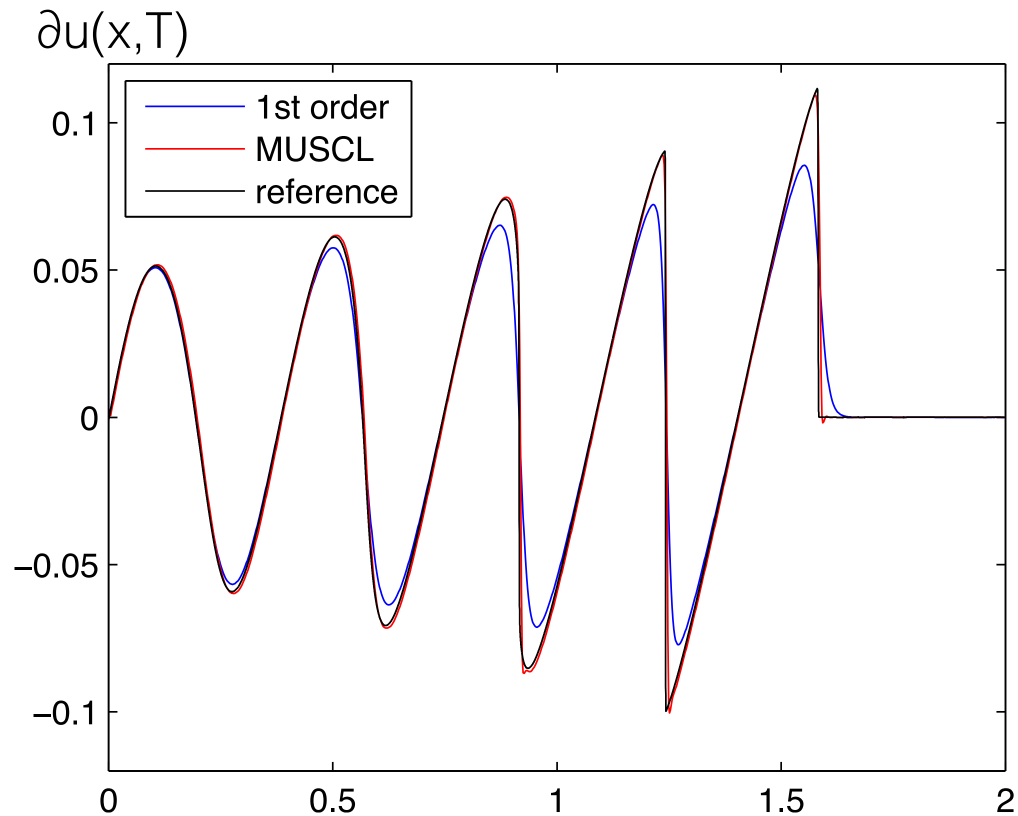


Numerical test: perturbation of an hydrostatic atmosphere

- Perfect gas law:
  $p = (\gamma - 1)\left(E - \rho u^2/2\right)$
- Constant gravitational field: $\phi(x) = gx$
- Steady state $w_s(x)$: hydrostatic atmosphere

$$\begin{cases} \rho_s(x) = \left(1 - \frac{\gamma-1}{\gamma} gx\right)^{\frac{1}{\gamma-1}} \\ u_s(x) = 0 \\ p_s(x) = \rho_s(x)^\gamma \end{cases}$$

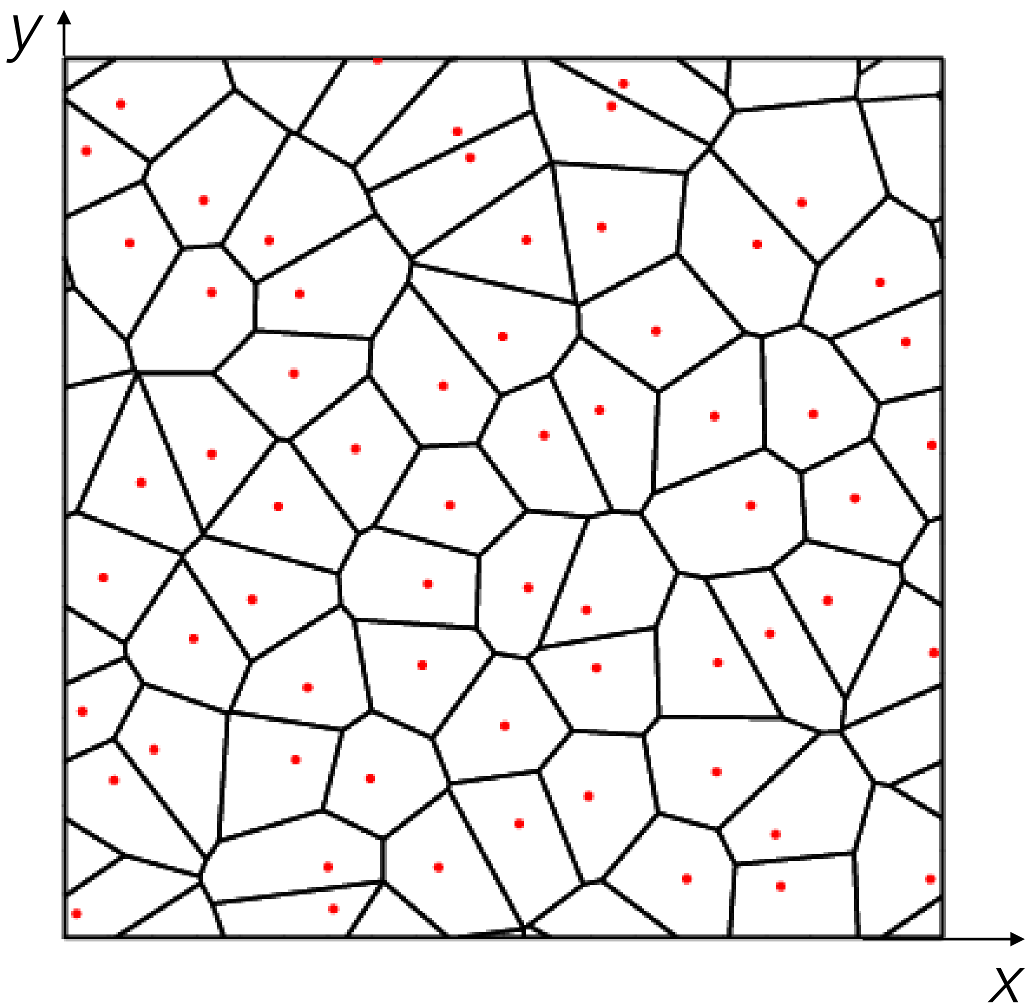
- Boundary condition:
  $u(0,t) = 0.1\sin(6\pi t)$
- Perturbation:
  $\delta w(x,t) = w(x,t) - w_s(x)$



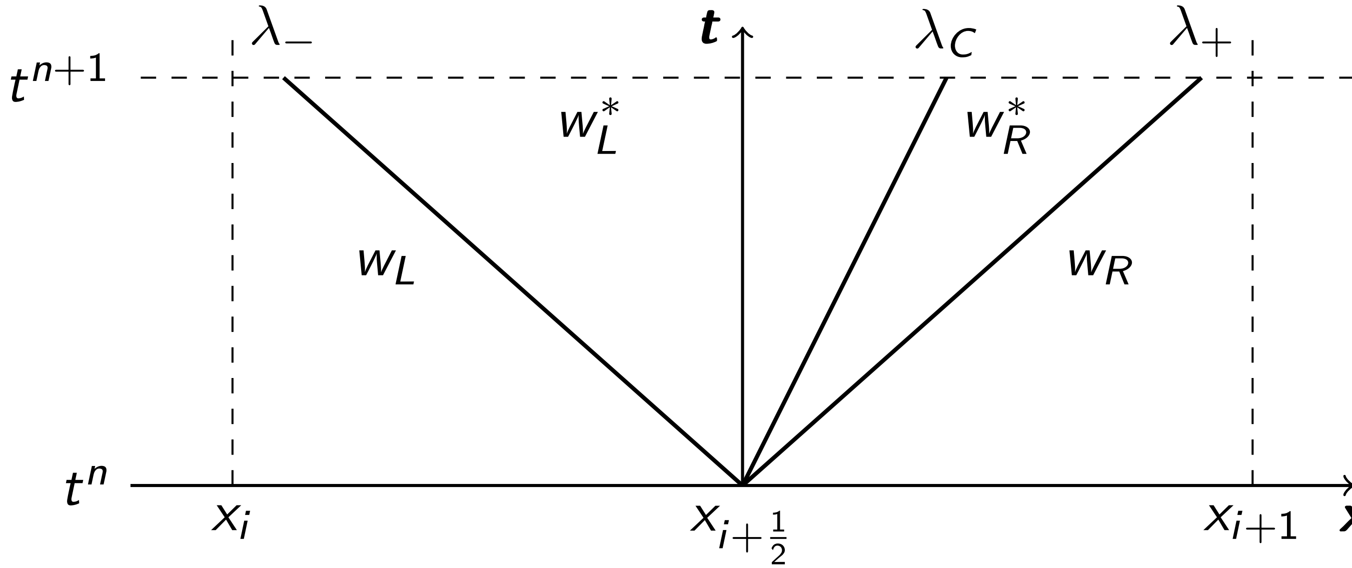
Final time perturbation in velocity $\delta u(x,T)$ computed with 1.024 cells.
The reference solution is computed with 32.768 cells with the first-order relaxation scheme.

jointly with Ohlmann, Röpke, Springel, Zenk:
“A second order well-balanced scheme on multi-dimensional unstructured grid”,
work in progress

The AREPO code



recall our relaxation well-balanced method:



$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0\\
(\rho u)_t + (\rho u^2 + \pi)_x &= -\rho \partial_x Z\\
(\rho E)_t + (u(\rho E + \pi))_x &= -\rho u \partial_x Z
\end{aligned}
$$

gravitational potential

The solution in the different regions is

$$
u^* = u_l^* = u_r^* = \frac{c_l u_l + c_r u_r}{c_l + c_r} + \frac{1}{c_l + c_r}(\pi_l - \pi_r + \overline{\rho}(\rho_l, \rho_r)(Z_l - Z_r))
$$

$$
\pi_l^* = \pi_l + c_l(u_l - u^*) \quad \pi_r^* = \pi_r + c_r(u^* - u_r)
$$

$$
\frac{1}{\rho_l^*} = \frac{1}{\rho_l} + \frac{1}{c_l}(u^* - u_l) \quad \frac{1}{\rho_r^*} = \frac{1}{\rho_r} + \frac{1}{c_r}(u_r - u^*)
$$

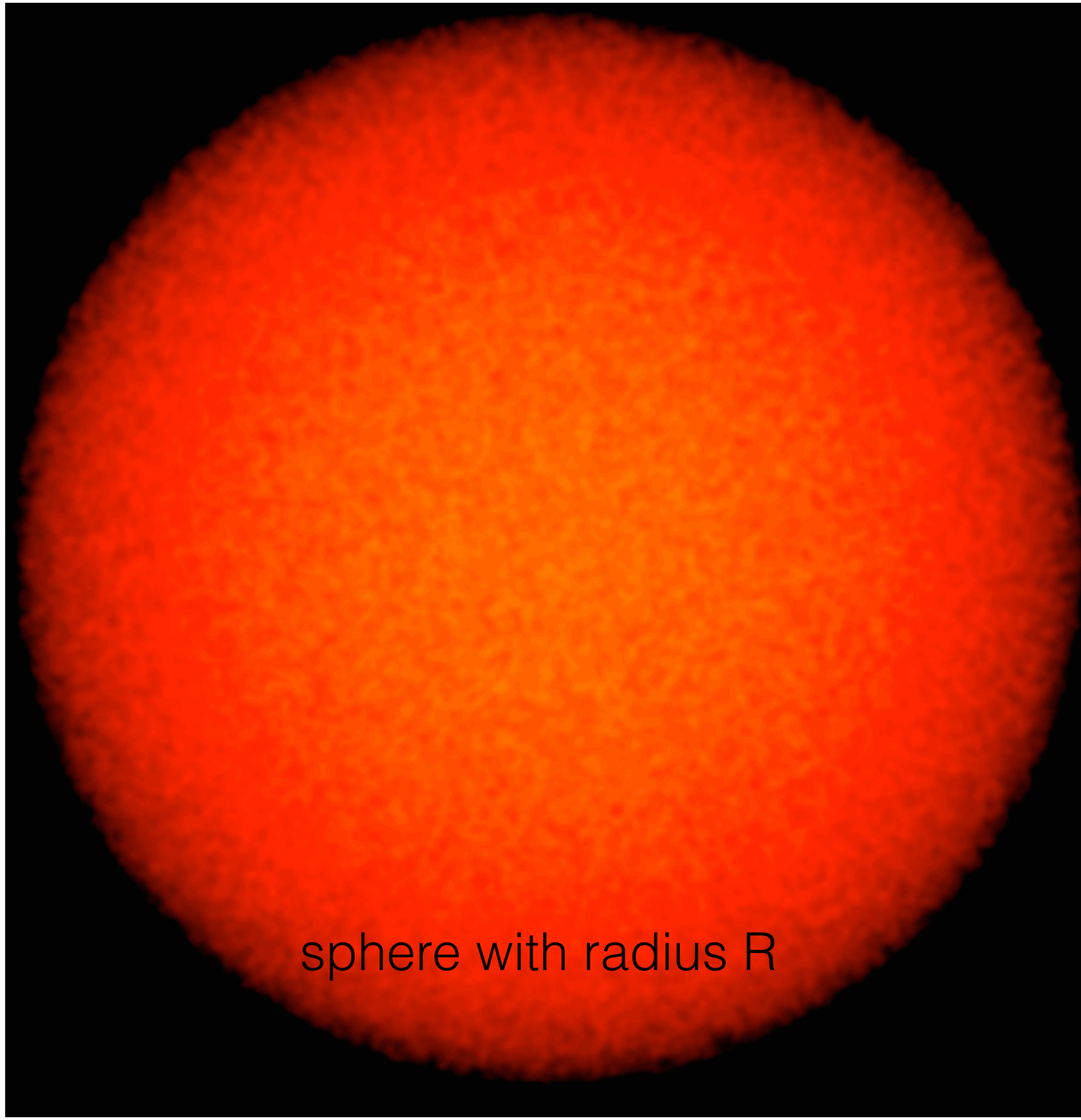
$$
e_l^* = e_l + \frac{1}{2c_l^2}((\pi_l^*)^2 - \pi_l^2)
$$

$$
e_r^* = e_r + \frac{1}{2c_r^2}((\pi_r^*)^2 - \pi_r^2).
$$

Evrard's collapse test

initial condition:

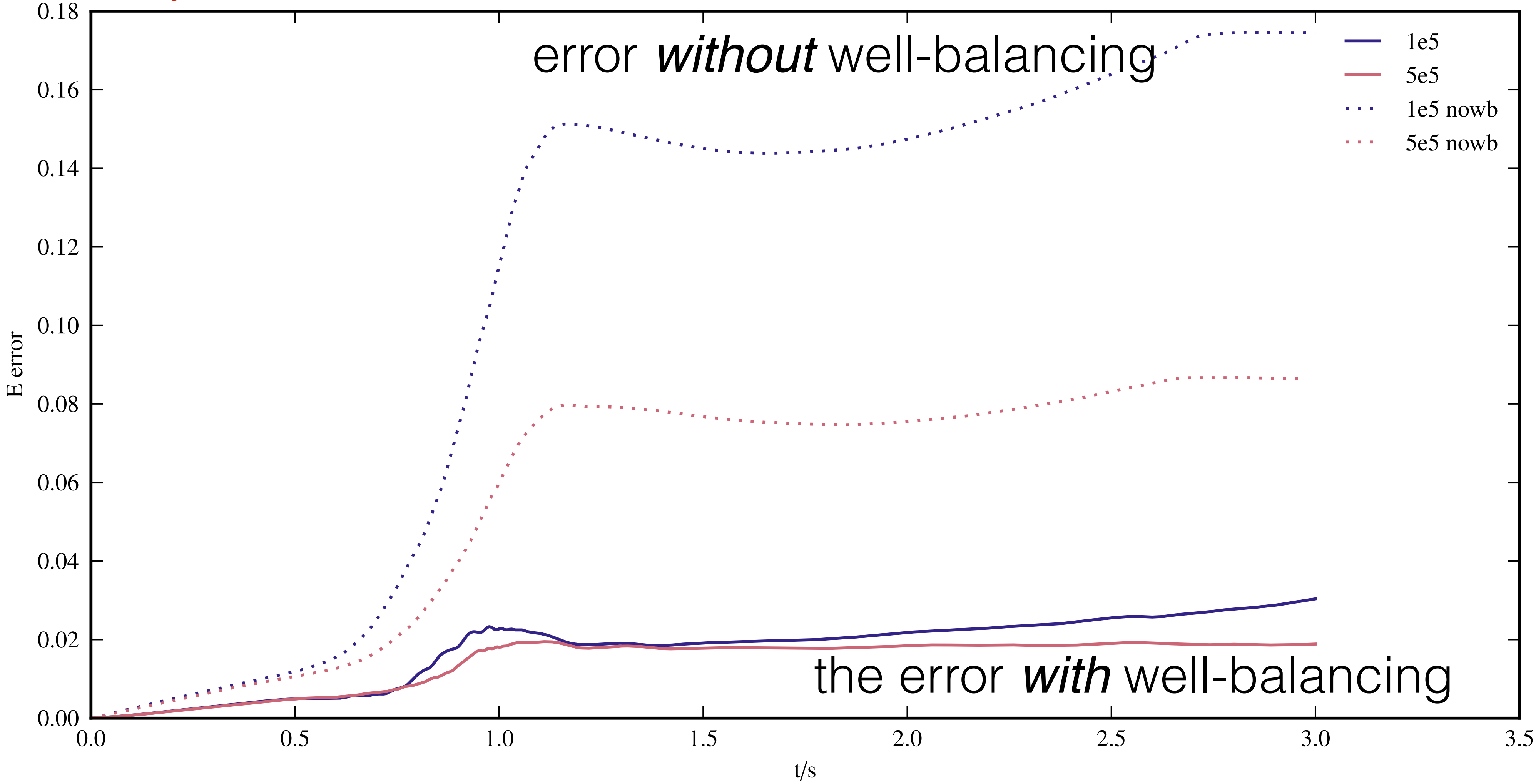
$$\rho(r) = \begin{cases} M/(2\pi R^2 r) & \text{for } r \leq R \\ 0 & \text{for } r > R. \end{cases}$$



this is a solution far away from hydrostatic equilibrium

Evrard's collapse test

$$\rho(r) = \begin{cases} M/(2\pi R^2 r) & \text{for } r \leq R \\ 0 & \text{for } r > R. \end{cases}$$



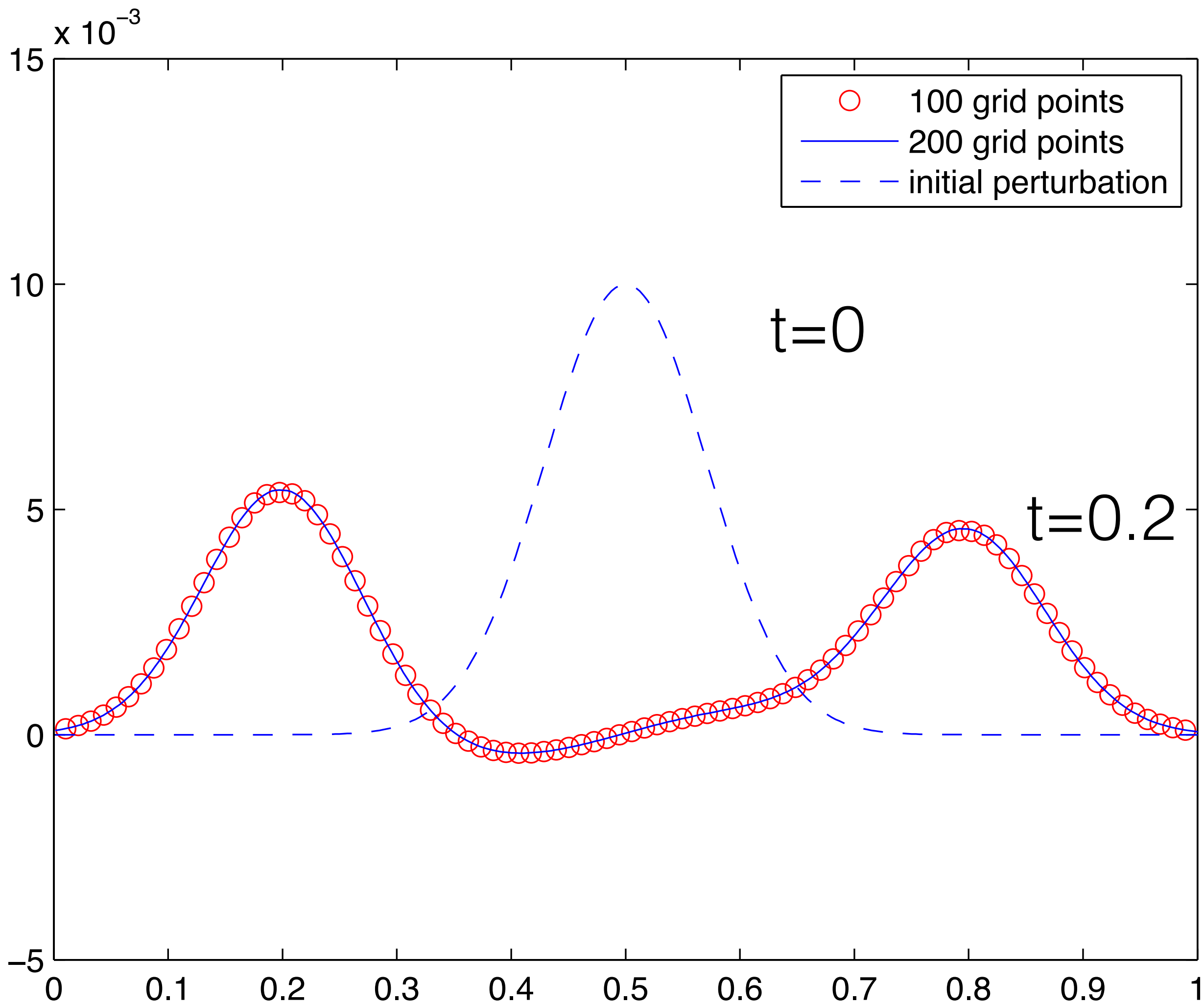
jointly with Ujjwal Koley and Rony Touma:

"Well-balanced unstaggered central schemes for the Euler equations with gravity",

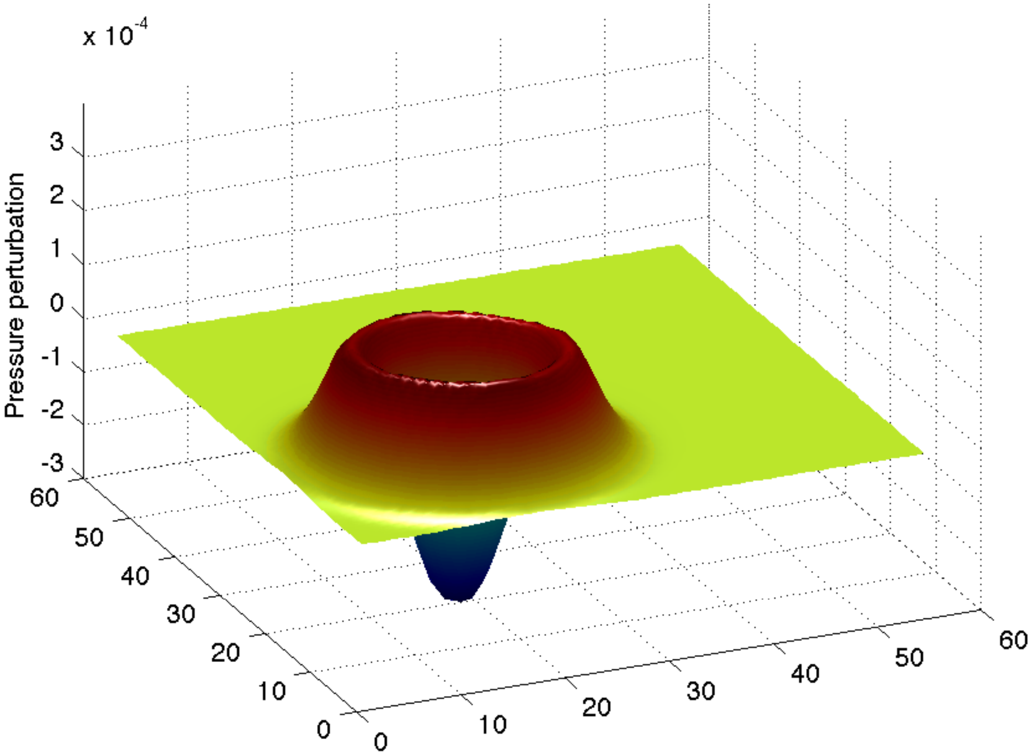
SIAM J. Sci. Comp (2015)

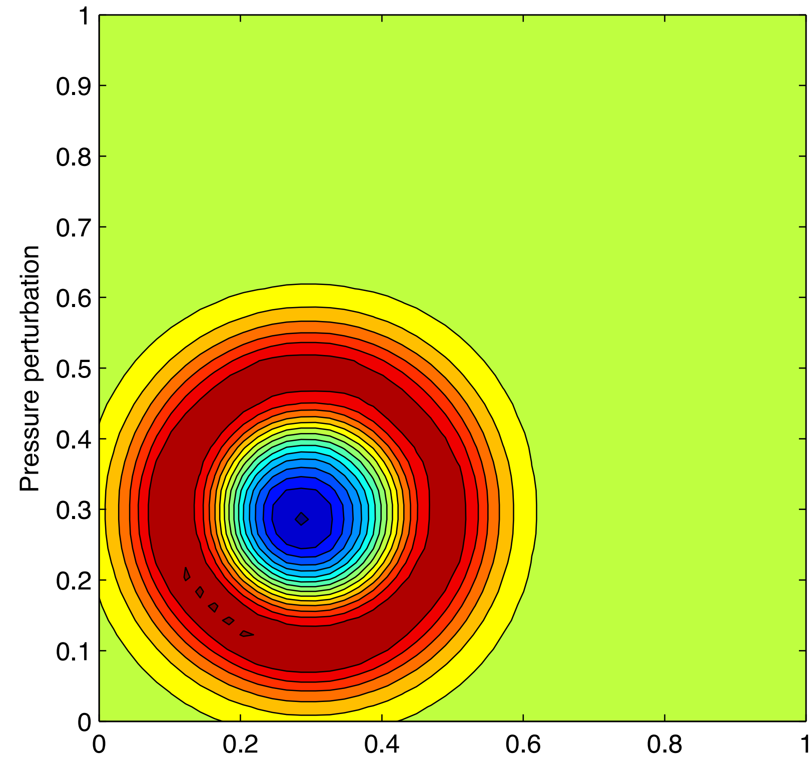
We construct a non-staggered central scheme, combine it with the "surface gradient method" in order to get a second order well-balanced scheme.

maintains specific equilibria: isothermal stationary solutions

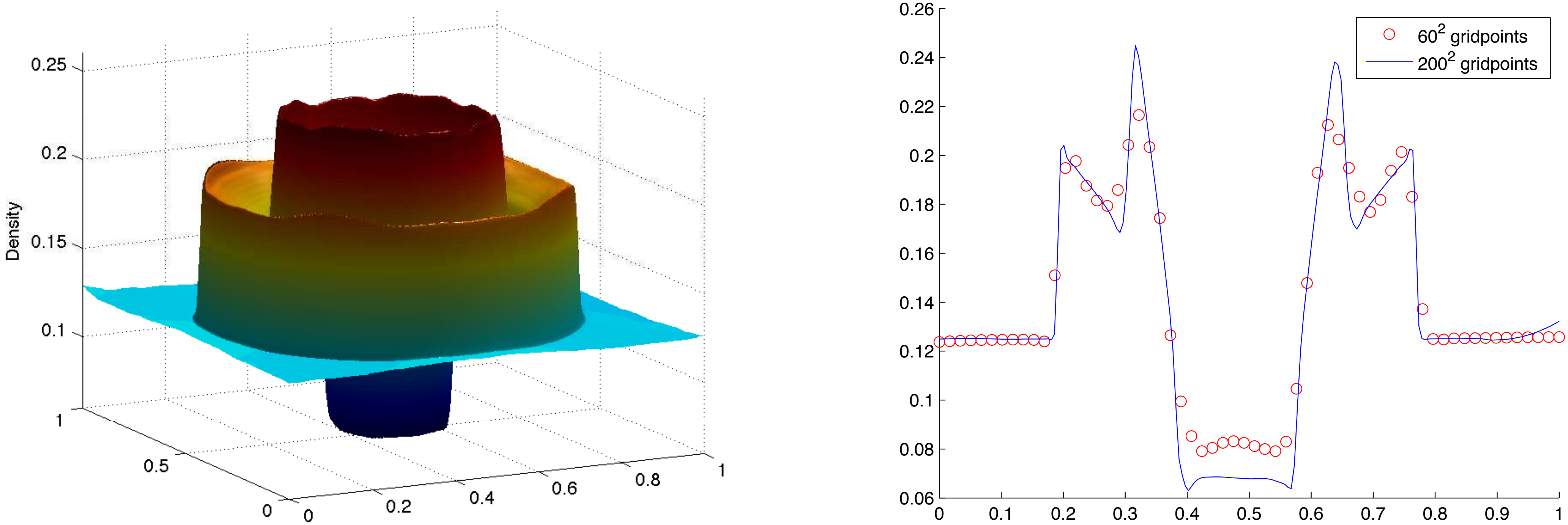


Perturbation of an equilibrium state; profile of the pressure disturbance at time t = 0.2





perturbation of an equilibrium state



circular Riemann problem: profile of the density

For the Euler equations ***with gravity***

$$
\begin{aligned}
&\rho_t + (\rho u)_x + (\rho v)_y = 0, \\
&(\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho u v)_y = -\rho \phi_x, \\
&(\rho v)_t + (\rho u v)_x + \left(\rho v^2 + p\right)_y = -\rho \phi_y, \\
&E_t + ((E+p)u)_x + ((E+p)v)_y = -\rho u \phi_x - \rho v \phi_y
\end{aligned}
$$

we found a ***well balanced methods.***

Consider flow that is a perturbation of the hydrostatic equilibrium, the flow is then close to incompressible flow.

Thus we need a solver that

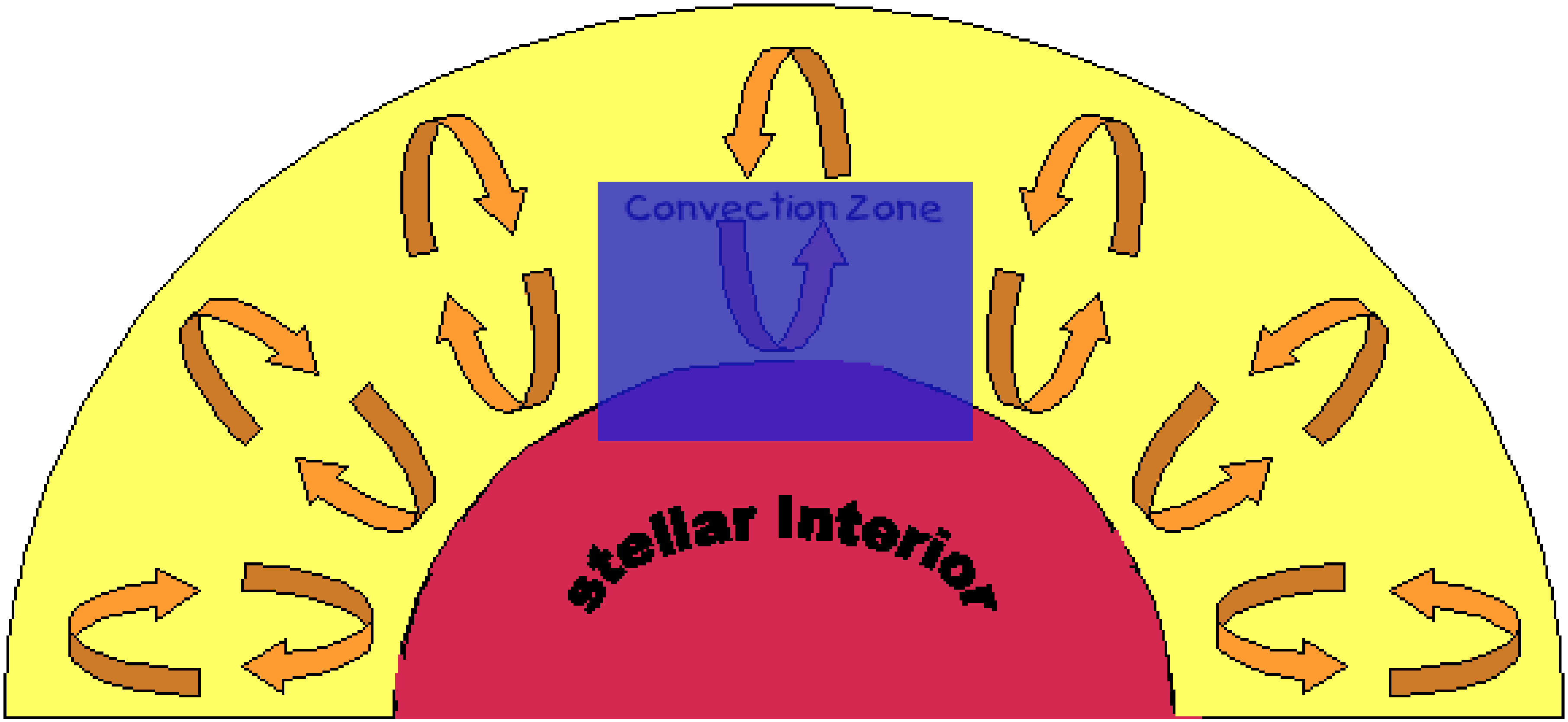
- *maintains hydrostatic equilibria well*
- *can solve low (as well as high) Mach number flow well.*

This is work in progress.

The finite volume well-balanced solver should in satisfy

- correct behavior of its dissipation matrix at low Mach numbers
- ensure kinetic energy is maintained near the incompressible regime
- be a linearly stable scheme
- the inversion of the large system of equations arising from implicit time discretization should have a good condition number

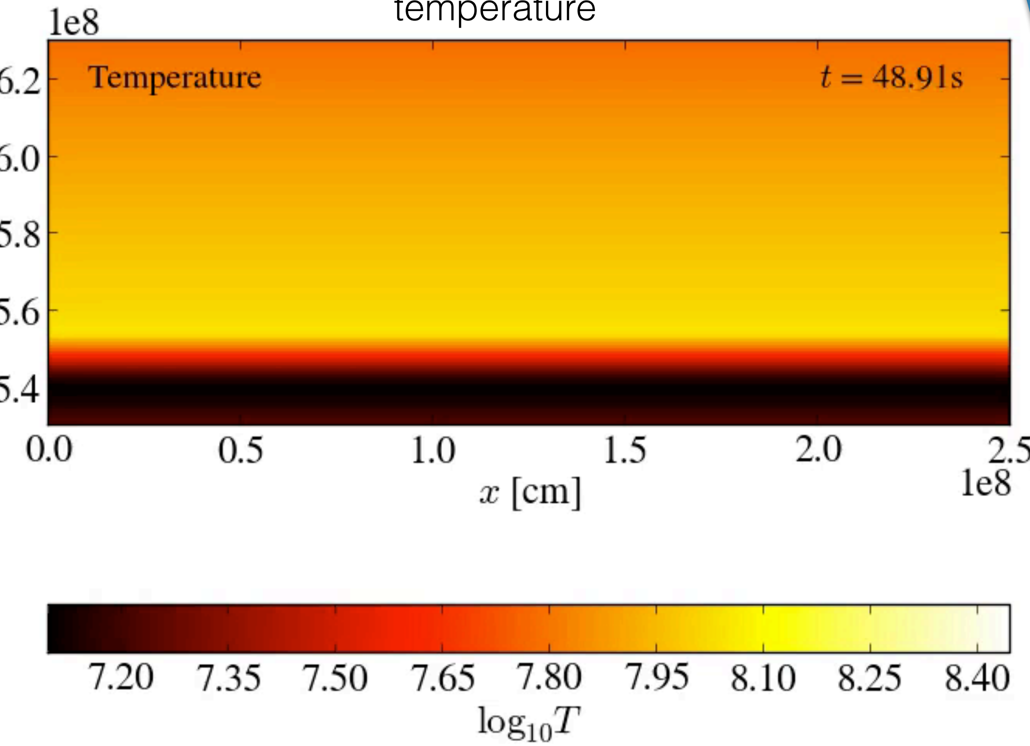
Numerical simulation of the code of the Fritz Röpke code for stellar convection.



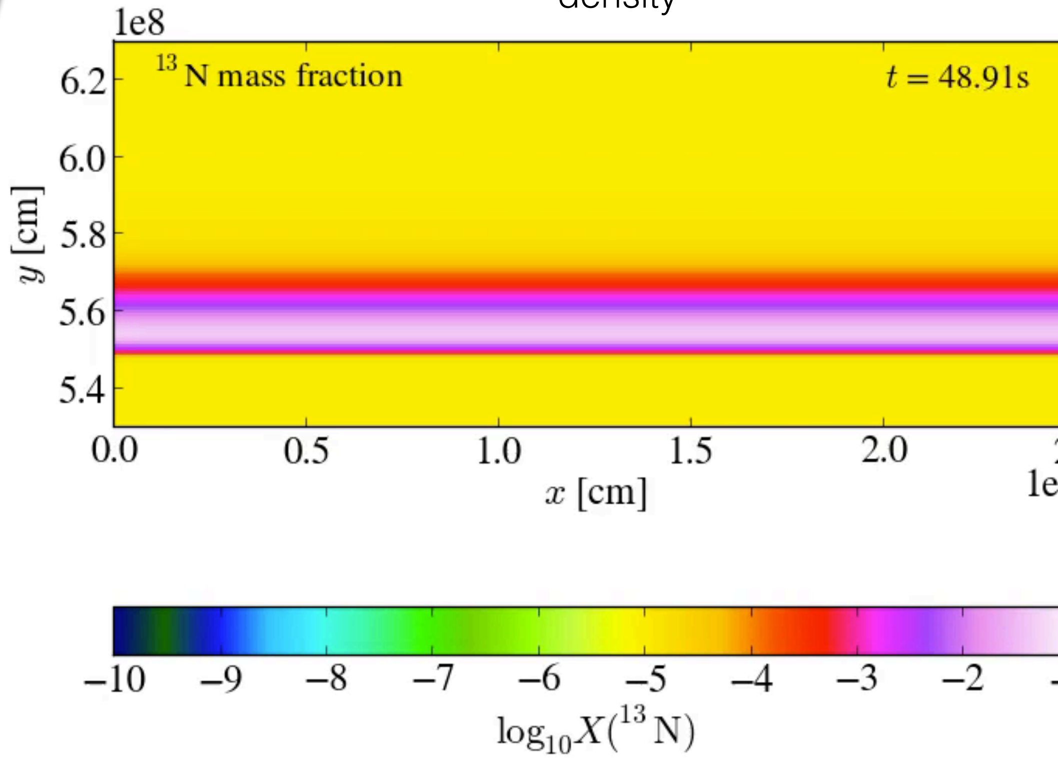
stellar atmosphere heated from below

gravity

temperature



density

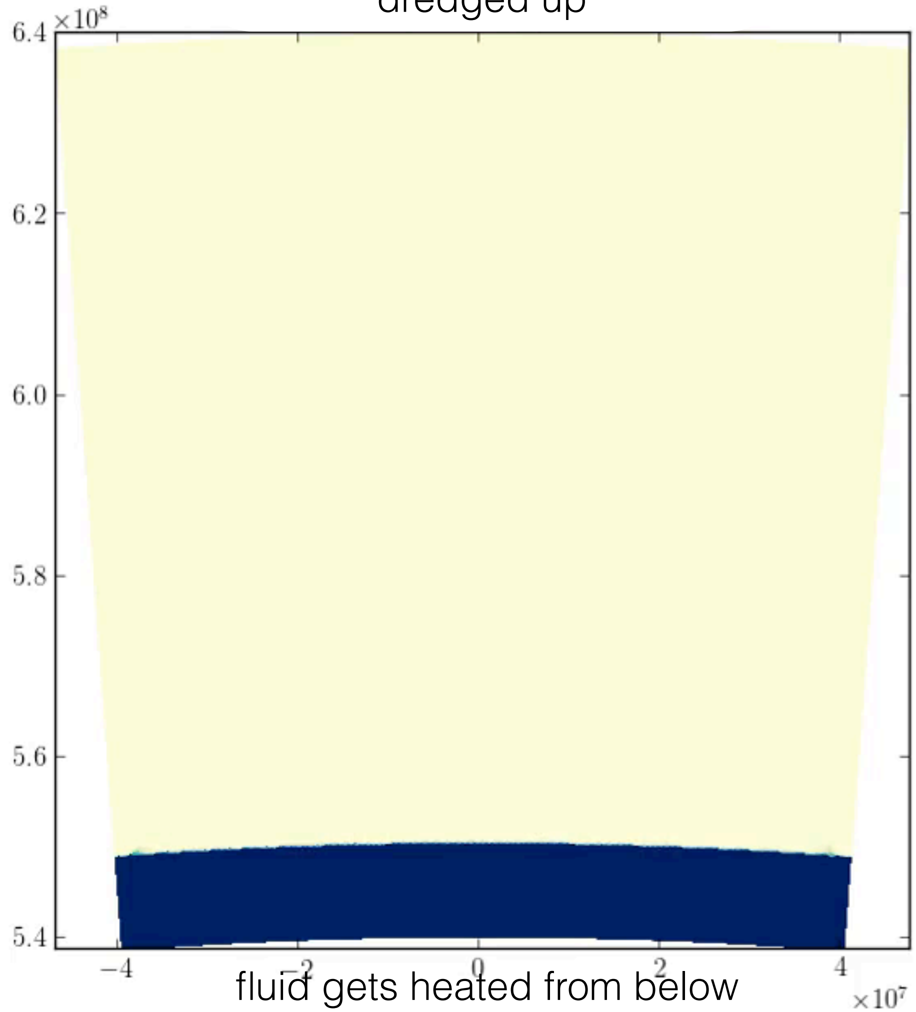


$t = 11.12\,\mathrm{s}$

density

gravity

how the bottom fluid gets
dredged up



fluid gets heated from below

Thank you for your attention!